Energy momentum tensor on lattice

Hiroshi Suzuki

Kyushu University

2017/11/30 @ KEK Tokai Campus

- H.S., Prog. Theor. Exp. Phys. (2013) 083B03 [arXiv:1304.0533 [hep-lat]]
- H. Makino and H.S., Prog. Theor. Exp. Phys. (2014) 063B02 [arXiv:1403.4772 [hep-lat]]
- M. Asakawa, T. Hatsuda, E. Itou, M. Kitazawa, H.S., Phys. Rev. D90 (2014) 011501 [arXiv:1312.7492 [hep-lat]]
- Y. Taniguchi, S. Ejiri, R. Iwami, K. Kanaya, M. Kitazawa, H.S., T. Umeda and N. Wakabayashi, Phys. Rev. D **96**, no. 1, 014509 (2017) [arXiv:1609.01417 [hep-lat]]

Gravity vs energy–momentum tensor (EMT)

- EMT $T_{\mu\nu}$ is a source of the Einstein gravity:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu},$$

where (assuming the Euclidean signature)

$$T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}}.$$

Gravity vs energy–momentum tensor (EMT)

- EMT $T_{\mu\nu}$ is a source of the Einstein gravity:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu},$$

where (assuming the Euclidean signature)

$$T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}}.$$

- For the gluon field,

$$S = \frac{1}{4g_0^2} \int d^4x\, \sqrt{g} g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho}^a F_{\nu\sigma}^a,$$

for instance, we have

$$T_{\mu\nu} = \frac{1}{g_0^2} \left(g^{\rho\sigma} F_{\mu\rho}^a F_{\nu\sigma}^a - \frac{1}{4} g_{\mu\nu} g^{\rho\sigma} g^{\lambda\tau} F_{\rho\lambda}^a F_{\sigma\tau}^a \right).$$

Another characterization of EMT

- In flat spacetime $g_{\mu\nu} \to \delta_{\mu\nu}$, EMT is the Noether current associated with the translational invariance:

$$\text{If} \qquad \delta S = 0 \qquad \text{under} \qquad \delta\varphi(x) = \xi_\mu \partial_\mu \varphi(x),$$

then

$$T_{\mu\nu}^{\text{canonical}} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \partial_\nu \varphi - \delta_{\mu\nu} \mathcal{L}, \qquad S = \int d^4x\, \mathcal{L},$$

is a conserved Noether current

$$\partial_\mu T_{\mu\nu}^{\text{canonical}} = 0.$$

Another characterization of EMT

- In flat spacetime $g_{\mu\nu} \to \delta_{\mu\nu}$, EMT is the Noether current associated with the translational invariance:

$$\text{If} \qquad \delta S = 0 \qquad \text{under} \qquad \delta\varphi(x) = \xi_\mu \partial_\mu \varphi(x),$$

then

$$T^{\text{canonical}}_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \partial_\nu \varphi - \delta_{\mu\nu} \mathcal{L}, \qquad S = \int d^4x\, \mathcal{L},$$

is a conserved Noether current

$$\partial_\mu T^{\text{canonical}}_{\mu\nu} = 0.$$

- Again, for the gluon field,

$$S = \frac{1}{4g_0^2} \int d^4x\, F^a_{\mu\nu} F^a_{\mu\nu},$$

we have

$$T^{\text{canonical}}_{\mu\nu} = \frac{1}{g_0^2} \left(F^a_{\mu\rho} \partial_\nu A^a_\rho - \frac{1}{4} \delta_{\mu\nu} F^a_{\rho\sigma} F^a_{\rho\sigma} \right).$$

Another characterization of EMT

- The above canonical EMT $T_{\mu\nu}^{\text{canonical}}$ is not quite identical to the symmetric (and gauge invariant) $T_{\mu\nu}$ even for $g_{\mu\nu} \to \delta_{\mu\nu}$, but this point can be remedied as

$$T_{\mu\nu} = T_{\mu\nu}^{\text{canonical}} - \frac{1}{g_0^2}\partial_\rho \left(F_{\mu\rho}^a A_\nu^a\right) + (\text{equation of motion}).$$

Another characterization of EMT

- The above canonical EMT $T_{\mu\nu}^{\text{canonical}}$ is not quite identical to the symmetric (and gauge invariant) $T_{\mu\nu}$ even for $g_{\mu\nu} \to \delta_{\mu\nu}$, but this point can be remedied as

$$T_{\mu\nu} = T_{\mu\nu}^{\text{canonical}} - \frac{1}{g_0^2}\partial_\rho\left(F_{\mu\rho}^a A_\nu^a\right) + (\text{equation of motion}).$$

- EMT is the conserved current associated with the translational invariance, a very fundamental observable.

Another characterization of EMT

- The above canonical EMT $T_{\mu\nu}^{\text{canonical}}$ is not quite identical to the symmetric (and gauge invariant) $T_{\mu\nu}$ even for $g_{\mu\nu} \to \delta_{\mu\nu}$, but this point can be remedied as

$$T_{\mu\nu} = T_{\mu\nu}^{\text{canonical}} - \frac{1}{g_0^2}\partial_\rho \left(F_{\mu\rho}^a A_\nu^a\right) + (\text{equation of motion}).$$

- EMT is the conserved current associated with the translational invariance, a very fundamental observable.
- Energy, momentum, angular-momentum, pressure, stress, viscosity, specific heat, renormalization group functions, ...

Another characterization of EMT

- The above canonical EMT $T_{\mu\nu}^{\text{canonical}}$ is not quite identical to the symmetric (and gauge invariant) $T_{\mu\nu}$ even for $g_{\mu\nu} \to \delta_{\mu\nu}$, but this point can be remedied as

$$T_{\mu\nu} = T_{\mu\nu}^{\text{canonical}} - \frac{1}{g_0^2}\partial_\rho \left(F_{\mu\rho}^a A_\nu^a\right) + (\text{equation of motion}).$$

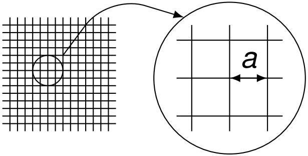
- EMT is the conserved current associated with the translational invariance, a very fundamental observable.
- Energy, momentum, angular-momentum, pressure, stress, viscosity, specific heat, renormalization group functions, ...
- We are interested in, for instance

$$\langle \text{baryon} | \, T_{\mu\nu} \, | \text{baryon} \rangle \, .$$

T_{00}: mass, T_{0i}: (angular-)momentum, (more ambitiously, coupling to the gravity).

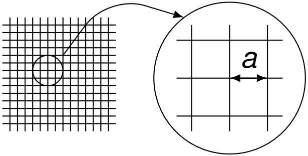
Non-perturbative calculation in gauge theories

- Best-understood approach is lattice regularization:



Non-perturbative calculation in gauge theories

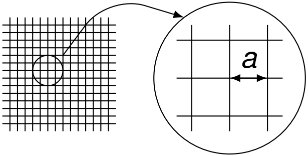
- Best-understood approach is lattice regularization:



- The lattice however breaks the translational invariance!

Non-perturbative calculation in gauge theories

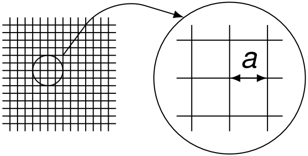
- Best-understood approach is lattice regularization:



- The lattice however breaks the translational invariance!
- No simple way to define EMT.

Non-perturbative calculation in gauge theories

- Best-understood approach is lattice regularization:



- The lattice however breaks the translational invariance!
- No simple way to define EMT.
- Naive lattice discretization of

$$T_{\mu\nu} = \frac{1}{g_0^2}\left(F^a_{\mu\rho}F^a_{\nu\rho} - \frac{1}{4}\delta_{\mu\nu}F^a_{\rho\sigma}F^a_{\rho\sigma}\right),$$

is neither correctly normalized nor conserved, even in the continuum limit $a \to 0$.

EMT in quantum field theory

- The origin of the trouble is that EMT

$$T_{\mu\nu} = \frac{1}{g_0^2}\left(F^a_{\mu\rho}F^a_{\nu\rho} - \frac{1}{4}\delta_{\mu\nu}F^a_{\rho\sigma}F^a_{\rho\sigma}\right)$$

is a local product (composite operator) of fields in QFT.

EMT in quantum field theory

- The origin of the trouble is that EMT

$$T_{\mu\nu} = \frac{1}{g_0^2}\left(F^a_{\mu\rho}F^a_{\nu\rho} - \frac{1}{4}\delta_{\mu\nu}F^a_{\rho\sigma}F^a_{\rho\sigma}\right)$$

  is a local product (composite operator) of fields in QFT.
- Propagator diverges as $x \to y$:

$$\langle\varphi(x)\varphi(y)\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-y)}}{k^2} \xrightarrow{x\to y} \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2} = \infty.$$

EMT in quantum field theory

- The origin of the trouble is that EMT

$$T_{\mu\nu} = \frac{1}{g_0^2}\left(F^a_{\mu\rho}F^a_{\nu\rho} - \frac{1}{4}\delta_{\mu\nu}F^a_{\rho\sigma}F^a_{\rho\sigma}\right)$$

is a local product (composite operator) of fields in QFT.

- Propagator diverges as $x \to y$:

$$\langle\varphi(x)\varphi(y)\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-y)}}{k^2} \xrightarrow{x\to y} \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2} = \infty.$$

- Then, something quite strange such as the trace anomaly,

$$T_{\mu\mu} = -\frac{\beta(g)}{2g^3}\{F_{\mu\nu}F_{\mu\nu}\}_R \neq 0 \qquad \Leftrightarrow \qquad \partial_\mu T_{\mu\nu} = 0$$

can occur ($0 \times \infty = 1$).

EMT in quantum field theory

- The origin of the trouble is that EMT

$$T_{\mu\nu} = \frac{1}{g_0^2}\left(F^a_{\mu\rho}F^a_{\nu\rho} - \frac{1}{4}\delta_{\mu\nu}F^a_{\rho\sigma}F^a_{\rho\sigma}\right)$$

  is a local product (composite operator) of fields in QFT.
- Propagator diverges as $x \to y$:

$$\langle\varphi(x)\varphi(y)\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-y)}}{k^2} \xrightarrow{x\to y} \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2} = \infty.$$

- Then, something quite strange such as the trace anomaly,

$$T_{\mu\mu} = -\frac{\beta(g)}{2g^3}\{F_{\mu\nu}F_{\mu\nu}\}_R \neq 0 \qquad \Leftrightarrow \qquad \partial_\mu T_{\mu\nu} = 0$$

  can occur ($0 \times \infty = 1$).
- More ingenious approach is necessary. . .

Gradient flow

- Gradient flow (Narayanan–Neuberger (2006), Lüscher (2010)) is an evolution of the gluon field along a fictitious time t; the initial value is the original gluon field:

$$B_\mu(t=0,x) = A_\mu(x).$$

The evolution for $t > 0$ is defined by

$$\partial_t B_\mu^a(t,x) = -g_0^2 \frac{\delta S}{\delta B_\mu^a(t,x)} = D_\nu G_{\nu\mu}^a(t,x) = \Delta B_\mu^a(t,x) + \cdots ,$$

where $D_\nu G_{\nu\mu}^a = \partial_\nu G_{\nu\mu}^a + f^{abc} B_\nu^b G_{\nu\mu}^c$ and
$G_{\mu\nu}^a = \partial_\mu B_\nu^a - \partial_\nu B_\mu^a + f^{abc} B_\mu^b B_\nu^c$.

Gradient flow

- Gradient flow (Narayanan–Neuberger (2006), Lüscher (2010)) is an evolution of the gluon field along a fictitious time t; the initial value is the original gluon field:

$$B_\mu(t=0,x)=A_\mu(x).$$

The evolution for $t>0$ is defined by

$$\partial_t B_\mu^a(t,x)=-g_0^2\frac{\delta S}{\delta B_\mu^a(t,x)}=D_\nu G_{\nu\mu}^a(t,x)=\Delta B_\mu^a(t,x)+\cdots,$$

where $D_\nu G_{\nu\mu}^a=\partial_\nu G_{\nu\mu}^a+f^{abc}B_\nu^b G_{\nu\mu}^c$ and $G_{\mu\nu}^a=\partial_\mu B_\nu^a-\partial_\nu B_\mu^a+f^{abc}B_\mu^b B_\nu^c$.

- A diffusion equation with the diffusion length $\sim\sqrt{8t}$.

Gradient flow

- Gradient flow (Narayanan–Neuberger (2006), Lüscher (2010)) is an evolution of the gluon field along a fictitious time t; the initial value is the original gluon field:

$$B_\mu(t = 0, x) = A_\mu(x).$$

  The evolution for $t > 0$ is defined by

$$\partial_t B_\mu^a(t, x) = -g_0^2 \frac{\delta S}{\delta B_\mu^a(t, x)} = D_\nu G_{\nu\mu}^a(t, x) = \Delta B_\mu^a(t, x) + \cdots ,$$

  where $D_\nu G_{\nu\mu}^a = \partial_\nu G_{\nu\mu}^a + f^{abc} B_\nu^b G_{\nu\mu}^c$ and $G_{\mu\nu}^a = \partial_\mu B_\nu^a - \partial_\nu B_\mu^a + f^{abc} B_\mu^b B_\nu^c$.
- A diffusion equation with the diffusion length $\sim \sqrt{8t}$.
- Any local product of flowed gluon fields B_μ is a renormalized finite operator (Lüscher–Weisz (2011)).

Gradient flow

- Gradient flow (Narayanan–Neuberger (2006), Lüscher (2010)) is an evolution of the gluon field along a fictitious time t; the initial value is the original gluon field:

$$B_\mu(t=0,x) = A_\mu(x).$$

The evolution for $t > 0$ is defined by

$$\partial_t B^a_\mu(t,x) = -g_0^2 \frac{\delta S}{\delta B^a_\mu(t,x)} = D_\nu G^a_{\nu\mu}(t,x) = \Delta B^a_\mu(t,x) + \cdots ,$$

where $D_\nu G^a_{\nu\mu} = \partial_\nu G^a_{\nu\mu} + f^{abc} B^b_\nu G^c_{\nu\mu}$ and $G^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + f^{abc} B^b_\mu B^c_\nu$.

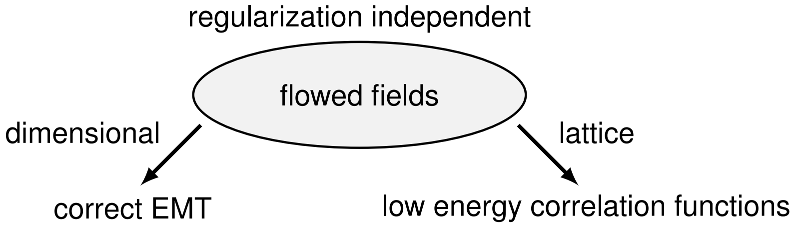
- A diffusion equation with the diffusion length $\sim \sqrt{8t}$.
- Any local product of flowed gluon fields B_μ is a renormalized finite operator (Lüscher–Weisz (2011)).
- Such a renormalized operator is independent of regularization

Our strategy

- We bridge lattice regularization and dimensional regularization which preserves the translational invariance, by using a flowed fields as an intermediate tool.

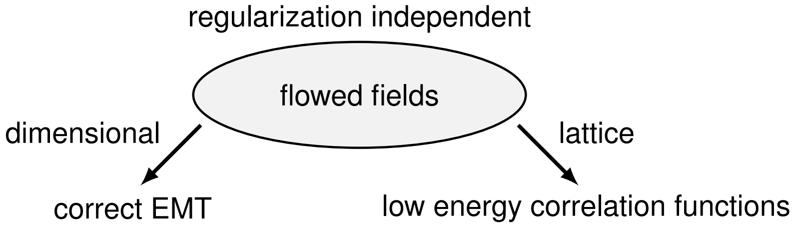
Our strategy

- We bridge lattice regularization and dimensional regularization which preserves the translational invariance, by using a flowed fields as an intermediate tool.
- Schematically,



Our strategy

- We bridge lattice regularization and dimensional regularization which preserves the translational invariance, by using a flowed fields as an intermediate tool.
- Schematically,



- Dimensional regularization,

$$4 \to D = 4 - 2\epsilon,$$

preserves both the gauge symmetry and the translational invariance, but, this is defined only in perturbation theory.

Regularization indep. expression of EMT (H.S. (2013), Makino–H.S. (2014))

- Universal formula

$$
\begin{aligned}
&T_{\mu\nu}(x)\\
&=\lim_{t\to 0}\Bigg\{c_1(t)G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)+\left[c_2(t)-\frac{1}{4}c_1(t)\right]\delta_{\mu\nu}G^a_{\rho\sigma}(t,x)G^a_{\rho\sigma}(t,x)\\
&\qquad+c_3(t)\mathring{\bar{\chi}}(t,x)\left(\gamma_\mu\overleftrightarrow{D}_\nu+\gamma_\nu\overleftrightarrow{D}_\mu\right)\mathring{\chi}(t,x)\\
&\qquad\qquad+\left[c_4(t)-2c_3(t)\right]\delta_{\mu\nu}\mathring{\bar{\chi}}(t,x)\overleftrightarrow{\not{D}}\mathring{\chi}(t,x)+c_5'(t)\mathring{\bar{\chi}}(t,x)\mathring{\chi}(t,x)-\mathrm{VEV}\Bigg\}
\end{aligned}
$$

where ($^{-}$ denotes running parameters in MS scheme)

$$c_1(t)=\frac{1}{\bar{g}(1/\sqrt{8t})^2}-b_0\ln\pi-\frac{1}{(4\pi)^2}\left[\frac{7}{3}C_2(G)-\frac{3}{2}T(R)N_{\mathrm{f}}\right],$$

$$c_2(t)=\frac{1}{8}\frac{1}{(4\pi)^2}\left[\frac{11}{3}C_2(G)+\frac{11}{3}T(R)N_{\mathrm{f}}\right],\quad c_3(t)=\frac{1}{4}\left\{1+\frac{\bar{g}(1/\sqrt{8t})^2}{(4\pi)^2}C_2(R)\left[\frac{3}{2}+\ln(432)\right]\right\},$$

$$c_4(t)=\frac{1}{8}d_0\bar{g}(1/\sqrt{8t})^2,\qquad c_5'(t)=-\bar{m}(1/\sqrt{8t})\left\{1+\frac{\bar{g}(1/\sqrt{8t})^2}{(4\pi)^2}C_2(R)\left[3\ln\pi+\frac{7}{2}+\ln(432)\right]\right\}$$

($b_0=\frac{1}{(4\pi)^2}\left[\frac{11}{3}C_2(G)-\frac{4}{3}T(R)N_{\mathrm{f}}\right]$ and $d_0=\frac{1}{(4\pi)^2}6C_2(R)$).)

Thermodynamic quantities in quenched QCD at finite temperature

- Asakawa–Hatsuda–Iritani–Itou–Kitazawa–H.S. (FlowQCD Collaboration)

Thermodynamic quantities in quenched QCD at finite temperature

- Asakawa–Hatsuda–Iritani–Itou–Kitazawa–H.S. (FlowQCD Collaboration)
- Thermodynamic quantities (trace anomaly, entropy density):

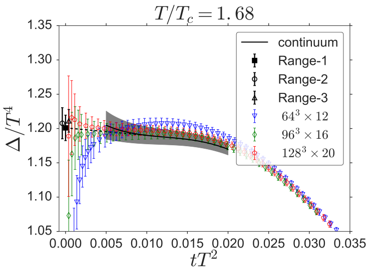
$$\langle \varepsilon - 3p \rangle = -\langle T_{\mu\mu} \rangle \,, \qquad \langle \varepsilon + p \rangle = -\langle T_{00} \rangle + \frac{1}{3} \sum_{i=1,2,3} \langle T_{ii} \rangle \,.$$

Thermodynamic quantities in quenched QCD at finite temperature

- Asakawa–Hatsuda–Iritani–Itou–Kitazawa–H.S. (FlowQCD Collaboration)
- Thermodynamic quantities (trace anomaly, entropy density):

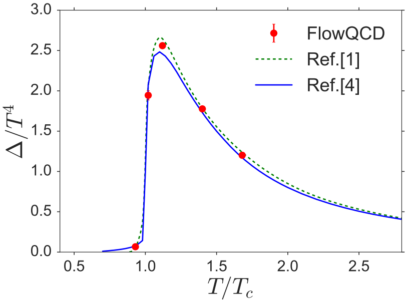
$$\langle \varepsilon - 3p \rangle = -\langle T_{\mu\mu} \rangle , \qquad \langle \varepsilon + p \rangle = -\langle T_{00} \rangle + \frac{1}{3} \sum_{i=1,2,3} \langle T_{ii} \rangle .$$

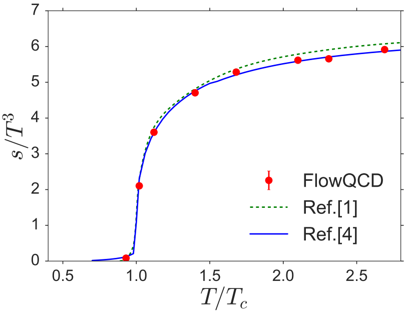
- $a = 0.013$–$0.061\,\mathrm{fm} \ll \sqrt{8t}$, number of configs. $\sim$ 1000–2000 :



Thermodynamic quantities in quenched QCD at finite temperature

- Trace anomaly and the entropy density

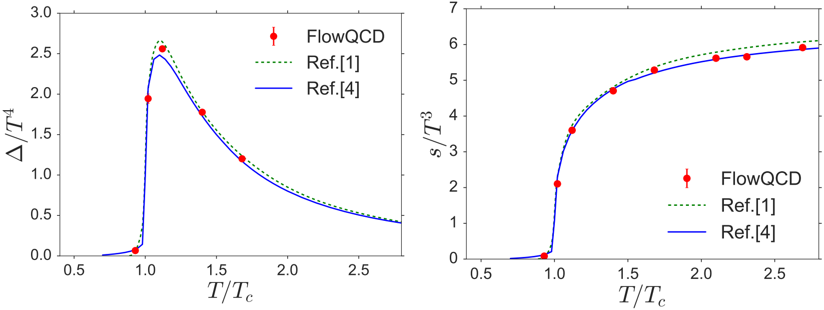




Ref. [1]: Boyd et al. (1996), Ref. [4]: Borsanyi et al. (2012), obtained by the integral method.

Thermodynamic quantities in quenched QCD at finite temperature

- Trace anomaly and the entropy density



Ref. [1]: Boyd et al. (1996), Ref. [4]: Borsanyi et al. (2012), obtained by the integral method.

- Works perfectly! No doubt on our reasoning.

Thermodynamic quantities in the $N_f = 2 + 1$ QCD

- Ejiri–Iwami–Kanaya–Kitazawa–H.S.–Taniguchi–Umeda–Wakabayashi [WHOT-QCD Collaboration]
- $a = 0.070\,\text{fm}$ fixed, $m_\pi/m_\rho \simeq 0.63$, $m_{\eta_{ss}}/m_\phi \simeq 0.74$, $N_s = 32$, number of configs. $\sim$ 100–1000.

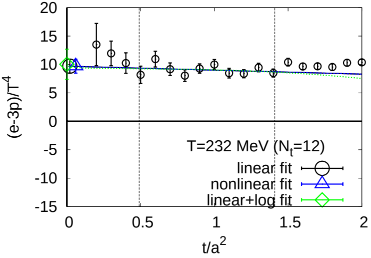


Figure: $(e - 3p)/T^4$, $T = 232\,\text{MeV}$

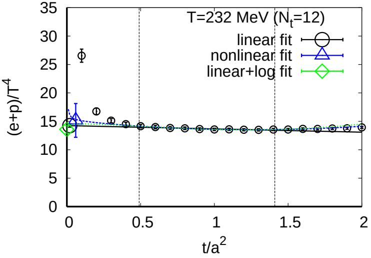


Figure: $(e + p)/T^4$, $T = 232\,\text{MeV}$

- $a = 0.070\,\mathrm{fm}$ fixed, $m_\pi/m_\rho \simeq 0.63$, $m_{\eta_{ss}}/m_\phi \simeq 0.74$, $N_s = 32$, number of configs. $\sim$ 100–1000.

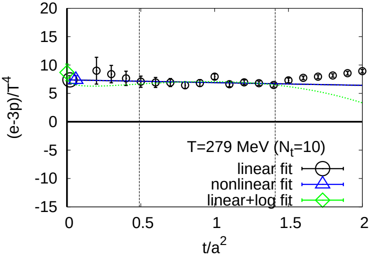


Figure: $(e-3p)/T^4$, $T = 279\,\mathrm{MeV}$

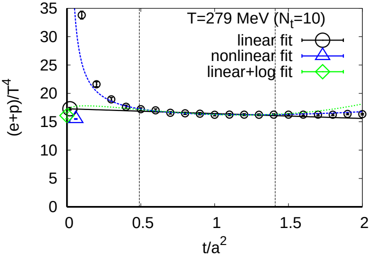


Figure: $(e+p)/T^4$, $T = 279\,\mathrm{MeV}$

Thermodynamic quantities in the $N_f = 2 + 1$ QCD

- $a = 0.070\,\mathrm{fm}$ fixed, $m_\pi/m_\rho \simeq 0.63$, $m_{\eta_{ss}}/m_\phi \simeq 0.74$, $N_s = 32$, number of configs. $\sim$ 100–1000.

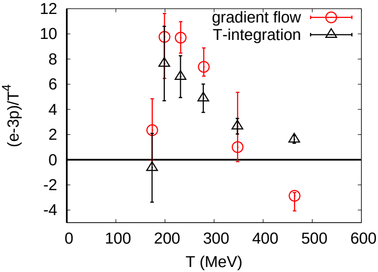


Figure: Black: T. Umeda et al. [WHOT-QCD Collaboration] (2012)

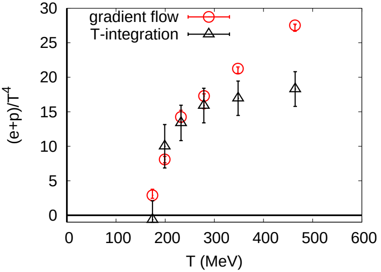


Figure: Black: T. Umeda et al. [WHOT-QCD Collaboration] (2012)

Summary

- We derived a regularization-independent representation of EMT in vector-like gauge theories through the gradient flow ⇒ Applicable to lattice gauge theory.

Summary

- We derived a regularization-independent representation of EMT in vector-like gauge theories through the gradient flow $\Rightarrow$ Applicable to lattice gauge theory.
- The 1 point function of EMT at finite temperature shows rather promising results.

Summary

- We derived a regularization-independent representation of EMT in vector-like gauge theories through the gradient flow $\Rightarrow$ Applicable to lattice gauge theory.
- The 1 point function of EMT at finite temperature shows rather promising results.
- We are now carrying out the computation with physical quark mass (K. Kanaya et al. [WHOT-QCD Collaboration], arXiv:1710.10015). Fairly good results.

Summary

- Recently, study of the 2 point correlation function

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(y) \rangle$$

has initiated (M. Kitazawa, T. Iritani, M. Asakawa, T. Hatsuda, arXiv:1708.01415; Y. Taniguchi et al. [WHOT-QCD Collaboration], arXiv:1711.02262) to examine the conservation law, linear response relations, the feasibility of the viscosity computation, etc.

Summary

- Recently, study of the 2 point correlation function

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(y) \rangle$$

  has initiated (M. Kitazawa, T. Iritani, M. Asakawa, T. Hatsuda, arXiv:1708.01415; Y. Taniguchi et al. [WHOT-QCD Collaboration], arXiv:1711.02262) to examine the conservation law, linear response relations, the feasibility of the viscosity computation, etc.
- The study of the hadronic matrix element

$$\langle \text{hadron} | T_{\mu\nu}(x) | \text{hadron} \rangle$$

  will also be interesting (ex. the spin structure).

Summary

- Recently, study of the 2 point correlation function

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(y) \rangle$$

  has initiated (M. Kitazawa, T. Iritani, M. Asakawa, T. Hatsuda, arXiv:1708.01415; Y. Taniguchi et al. [WHOT-QCD Collaboration], arXiv:1711.02262) to examine the conservation law, linear response relations, the feasibility of the viscosity computation, etc.
- The study of the hadronic matrix element

$$\langle \text{hadron} | \, T_{\mu\nu}(x) \, | \text{hadron} \rangle$$

  will also be interesting (ex. the spin structure).
- Also for the gravitational physics?