Workshop on Progress on Hadron Structure Functions in 2018

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Introduction to Quantum Entanglement

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1. What is quantum entanglement?

For a system consisting two subsystems (bipartite system)

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$$

any quantum (pure) state $|\Psi_{AB}\rangle \in \mathcal{H}_{AB}$ can be written as

$$|\Psi_{AB}\rangle = \sum_{i=0}^{d_A-1} \sum_{j=0}^{d_B-1} A_{ij}^{\Psi} |e_A^i\rangle \otimes |e_B^j\rangle$$

ONB of $\mathcal{H}_A$ ONB of $\mathcal{H}_B$

If $|\Psi_{AB}\rangle = |\phi_A\rangle|\psi_B\rangle$ → Separable

If $|\Psi_{AB}\rangle \neq |\phi_A\rangle|\psi_B\rangle$ → Entangled

- Examples: two qubits system $d_A = 2 \quad d_B = 2$

spin singlet state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)$$

or

Bell states

$$|\psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|0\rangle\,|1\rangle \pm |1\rangle\,|0\rangle) \qquad |\phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|0\rangle\,|0\rangle \pm |1\rangle\,|1\rangle)$$

> ... the best possible knowledge of a whole does not necessarily include the best possible knowledge of all its parts, even though they may be entirely separate ...
>
> I would not call that one but rather the characteristic trait of quantum mechanics

Schrödinger
1935

- Examples: three qubits system

GHZ (Greenberger, Horne and Zeilinger) state

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

W state

$$|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$$

- Examples: two bosonic system

NOON state

mode number (n photons)

$$|\mathrm{NOON}\rangle = \frac{1}{\sqrt{2}}(|n\rangle|0\rangle + |0\rangle|n\rangle)$$

For a mixed state,

If

$$\varrho_{AB} = \sum_{i=1}^{k} p_i \varrho_A^i \otimes \varrho_B^i, \longrightarrow \text{Separable}$$

ϱ_A^i ↑ $\mathcal{H}_A$ ϱ_B^i ↑ $\mathcal{H}_B$

If

$$\varrho_{AB} \neq \sum_{i=1}^{k} p_i \varrho_A^i \otimes \varrho_B^i, \longrightarrow \text{Entangled}$$

2. Ontological question: *EPR*

of lanthanum is 7/2, hence the nuclear magnetic moment as determined by this analysis is 2.5 nuclear magnetons. This is in fair agreement with the value 2.8 nuclear magnetons determined from La III hyperfine structures by the writer and N. S. Grace.[9]

[9] M. F. Crawford and N. S. Grace, Phys. Rev. **47**, 536 (1935).

This investigation was carried out under the supervision of Professor G. Breit, and I wish to thank him for the invaluable advice and assistance so freely given. I also take this opportunity to acknowledge the award of a Fellowship by the Royal Society of Canada, and to thank the University of Wisconsin and the Department of Physics for the privilege of working here.

MAY 15, 1935 PHYSICAL REVIEW VOLUME 47

Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

A. EINSTEIN, B. PODOLSKY AND N. ROSEN, *Institute for Advanced Study, Princeton, New Jersey*
(Received March 25, 1935)

In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system. In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration of the problem of making predictions concerning a system on the basis of measurements made on another system that had previously interacted with it leads to the result that if (1) is false then (2) is also false. One is thus led to conclude that the description of reality as given by a wave function is not complete.

1.

ANY serious consideration of a physical theory must take into account the distinction between the objective reality, which is independent of any theory, and the physical concepts with which the theory operates. These concepts are intended to correspond with the objective reality, and by means of these concepts we picture this reality to ourselves.

In attempting to judge the success of a physical theory, we may ask ourselves two questions: (1) "Is the theory correct?" and (2) "Is the description given by the theory complete?" It is only in the case in which positive answers may be given to both of these questions, that the concepts of the theory may be said to be satisfactory. The correctness of the theory is judged by the degree of agreement between the conclusions of the theory and human experience. This experience, which alone enables us to make inferences about reality, in physics takes the form of experiment and measurement. It is the second question that we wish to consider here, as applied to quantum mechanics.

Whatever the meaning assigned to the term *complete*, the following requirement for a complete theory seems to be a necessary one: *every element of the physical reality must have a counterpart in the physical theory*. We shall call this the condition of completeness. The second question is thus easily answered, as soon as we are able to decide what are the elements of the physical reality.

The elements of the physical reality cannot be determined by *a priori* philosophical considerations, but must be found by an appeal to results of experiments and measurements. A comprehensive definition of reality is, however, unnecessary for our purpose. We shall be satisfied with the following criterion, which we regard as reasonable. *If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.* It seems to us that this criterion, while far from exhausting all possible ways of recognizing a physical reality, at least provides us with one

Criterion of Completeness

Every element of the physical reality must have a counterpart in the physical theory.

Einstein-Podolsky-Rosen (1935)

Criterion of Reality

If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of reality corresponding to that quantity.

example: entangled state of two electrons (spin singlet)

$$|\psi\rangle = |+z\rangle|-z\rangle - |-z\rangle|+z\rangle = |+x\rangle|-x\rangle - |-x\rangle|+x\rangle$$

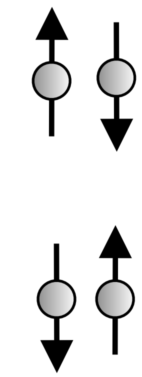
rotational symmetry

perfect correlation

$$|\psi\rangle = |+z\rangle|-z\rangle + |-z\rangle|+z\rangle$$

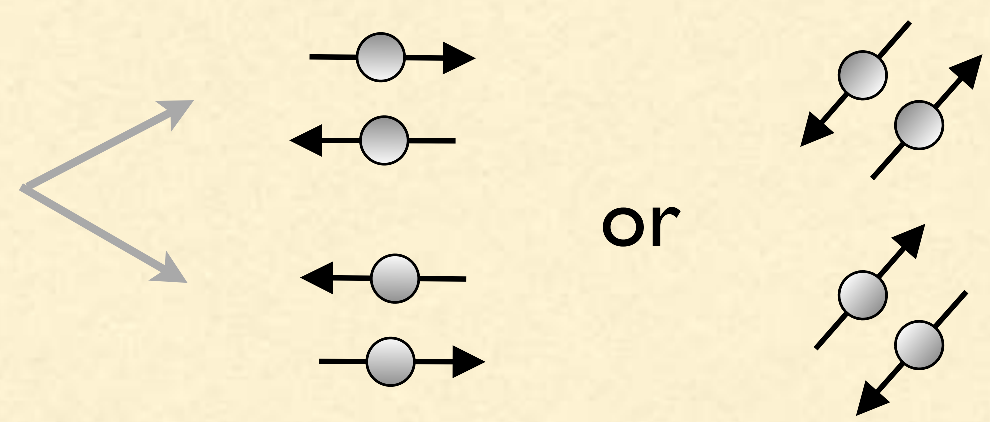
$|+z\rangle|-z\rangle$

$|-z\rangle|+z\rangle$

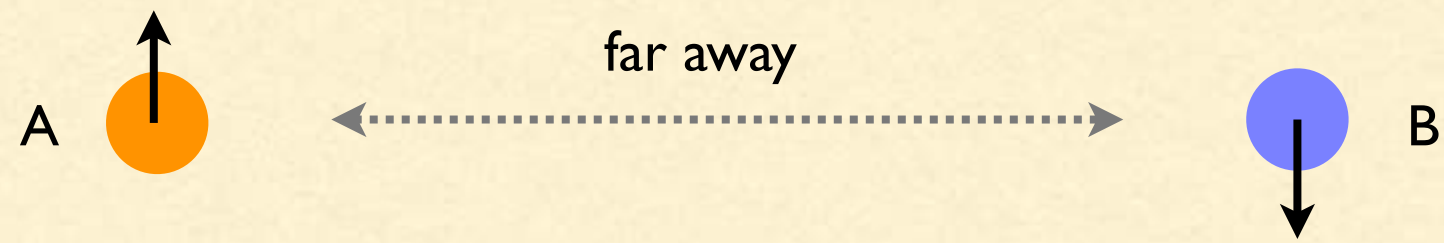


measurement of one electron

because of the rotational symmetry of the singlet state, we can measure the spin in any direction (component) we like and still observe the same perfect correlation



if we separate the two electrons far apart

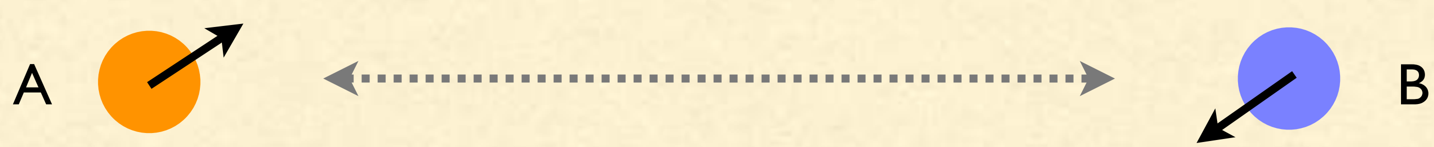


measuring A will not affect B → B will not be disturbed

locality

value of A → (correlation) value of B ······ reality of the value of B

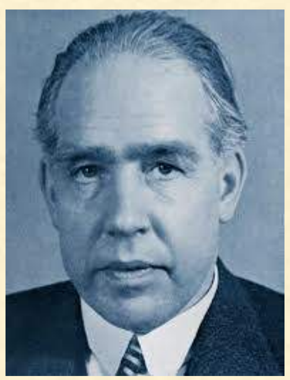
but we can choose any component for measurement of A



→ any spin component of B must possess physical reality but QM does not allow two components of spin determined simultaneously

→ therefore, QM is not complete as a physical theory!

Bohr's objection (1935)



reality of physical quantities can be discussed only when they can be observed simultaneously

since different spin components cannot be measured simultaneously, they cannot be discussed as physical reality in the same experimental context

context determines physical reality!

OCTOBER 15, 1935 PHYSICAL REVIEW VOLUME 48

Can Quantum-Mechanical Description of Physical Reality be Considered Complete?

N. Bohr, *Institute for Theoretical Physics, University, Copenhagen*

(Received July 13, 1935)

It is shown that a certain "criterion of physical reality" formulated in a recent article with the above title by A. Einstein, B. Podolsky and N. Rosen contains an essential ambiguity when it is applied to quantum phenomena. In this connection a viewpoint termed "complementarity" is explained from which quantum-mechanical description of physical phenomena would seem to fulfill, within its scope, all rational demands of completeness.

IN a recent article[1] under the above title A. Einstein, B. Podolsky and N. Rosen have presented arguments which lead them to answer the question at issue in the negative. The trend of their argumentation, however, does not seem to me adequately to meet the actual situation with which we are faced in atomic physics. I shall therefore be glad to use this opportunity to explain in somewhat greater detail a general viewpoint, conveniently termed "complementarity," which I have indicated on various previous occasions,[2] and from which quantum mechanics within its scope would appear as a completely rational description of physical phenomena, such as we meet in atomic processes.

The extent to which an unambiguous meaning can be attributed to such an expression as "physical reality" cannot of course be deduced from *a priori* philosophical conceptions, but—as the authors of the article cited themselves emphasize—must be founded on a direct appeal to experiments and measurements. For this purpose they propose a "criterion of reality" formulated as follows: "If, without in any way disturbing a system, we can predict with certainty the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity." By means of an interesting example, to which we shall return below, they next proceed to show that in quantum mechanics, just as in classical mechanics, it is possible under suitable conditions to predict the value of any given variable pertaining to the description of a mechanical system from measurements performed entirely on other systems which previously have been in interaction with the system under investigation. According to their criterion the authors therefore want to ascribe an element of reality to each of the quantities represented by such variables. Since, moreover, it is a well-known feature of the present formalism of quantum mechanics that it is never possible, in the description of the state of a mechanical system, to attach definite values to both of two canonically conjugate variables, they consequently deem this formalism to be incomplete, and express the belief that a more satisfactory theory can be developed.

Such an argumentation, however, would hardly seem suited to affect the soundness of quantum-mechanical description, which is based on a coherent mathematical formalism covering automatically any procedure of measurement like that indicated.* The apparent contradiction in

[1] A. Einstein, B. Podolsky and N. Rosen, Phys. Rev. **47**, 777 (1935).

[2] Cf. N. Bohr, *Atomic Theory and Description of Nature*, I (Cambridge, 1934).

* The deductions contained in the article cited may in this respect be considered as an immediate consequence of the transformation theorems of quantum mechanics, which perhaps more than any other feature of the formalism contribute to secure its mathematical completeness and its rational correspondence with classical mechanics. In fact, it is always possible in the description of a mechanical system, consisting of two partial systems (1) and (2), interacting or not, to replace any two pairs of canonically conjugate variables (q_1p_1), (q_2p_2) pertaining to systems (1) and (2), respectively, and satisfying the usual commutation rules

$$[q_1p_1]=[q_2p_2]=ih/2\pi,$$
$$[q_1q_2]=[p_1p_2]=[q_1p_2]=[q_2p_1]=0,$$

by two pairs of new conjugate variables (Q_1P_1), (Q_2P_2) related to the first variables by a simple orthogonal transformation, corresponding to a rotation of angle θ in the planes (q_1q_2), (p_1p_2)

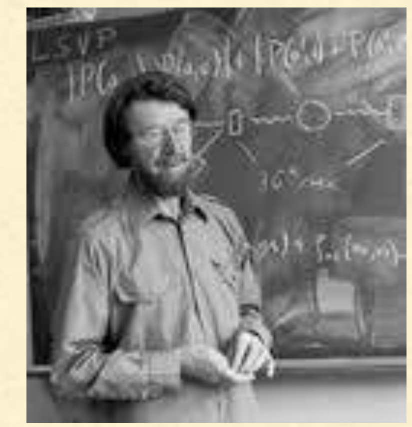
$$q_1=Q_1\cos\theta-Q_2\sin\theta \qquad p_1=P_1\cos\theta-P_2\sin\theta$$
$$q_2=Q_1\sin\theta+Q_2\cos\theta \qquad p_2=P_1\sin\theta+P_2\cos\theta.$$

Since these variables will satisfy analogous commutation rules, in particular

$$[Q_1P_1]=ih/2\pi, \qquad [Q_1P_2]=0,$$

it follows that in the description of the state of the combined system definite numerical values may not be assigned to both Q_1 and P_1, but that we may clearly assign

3. Quantum correlation: *Bell's inequality*



Bell (1928 - 1990)

Is local realism admissible experimentally?

Reality

$A(a, b, B, \lambda)$ $B(a, b, A, \lambda)$

external parameters (a, b) · hidden variable (λ) → probability distribution $\rho(\lambda)$

Locality

$A(a, b, B, \lambda) = A(a, b, \lambda)$ outcome independence

$A(a, b, \lambda) = A(a, \lambda)$ parameter independence

correlations in local realistic theory

(= our classical world)

measurement outcome $\quad A(a,\lambda) = \pm 1 \qquad B(b,\lambda) = \pm 1$

correlation $$C(a,b) = \int d\lambda \, \rho(\lambda) \, A(a,\lambda) \, B(b,\lambda)$$

probability distribution

combination of correlations

Bell (1964)

$$|C(a,b) - C(a,b')| + |C(a',b') + C(a',b)| \leq 2$$

Bell (CHSH) inequality

Proof:

$$C(a,b) - C(a,b') = \int d\lambda\, \rho(\lambda)\, [A(a,\lambda)\, B(b,\lambda) - A(a,\lambda)\, B(b',\lambda)]$$

$$= \int d\lambda\, \rho(\lambda)\, A(a,\lambda)\, B(b,\lambda)\, [1 \pm A(a',\lambda)\, B(b',\lambda)]$$

$$- \int d\lambda\, \rho(\lambda)\, A(a,\lambda)\, B(b',\lambda)\, [1 \pm A(a',\lambda)\, B(b,\lambda)]$$

from triangular inequality

$$|C(a,b) - C(a,b')| \leq \int d\lambda\, \rho(\lambda)\, [1 \pm A(a',\lambda)\, B(b',\lambda)]$$

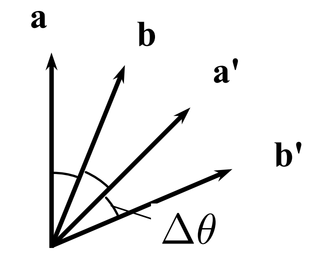
$$+ \int d\lambda\, \rho(\lambda)\, [1 \pm A(a',\lambda)\, B(b,\lambda)]$$

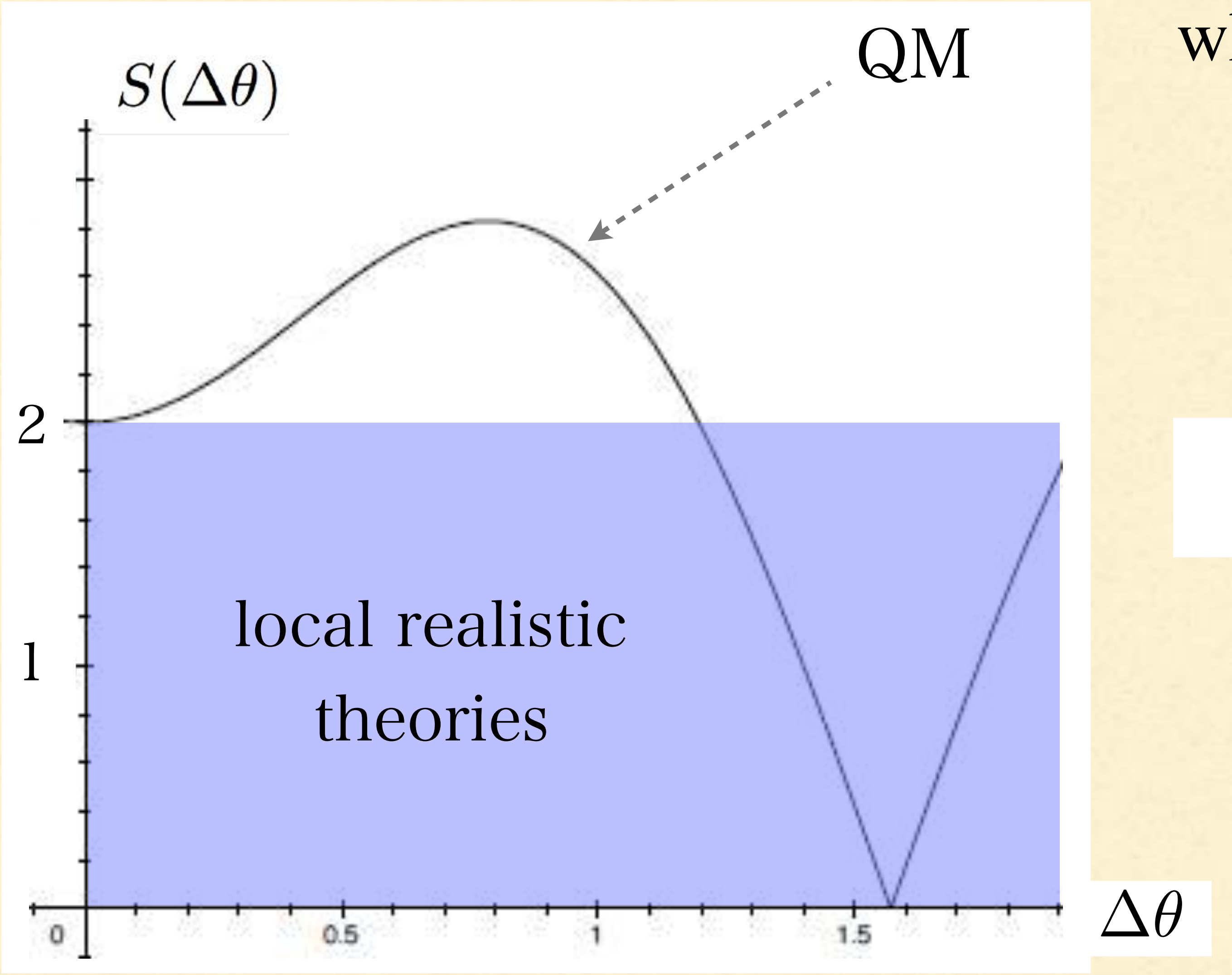
$$= 2 \pm [C(a',b') + C(a',b)]$$

$$\longrightarrow \quad |C(a,b) - C(a,b')| + |C(a',b') + C(a',b)| \leq 2$$

quantum mechanically

$$S(\Delta\theta) = |C(a,b) - C(a,b')| + |C(a',b') + C(a',b)|$$





while in QM:

$$C(a,b) \to -\cos(\Delta\theta)$$

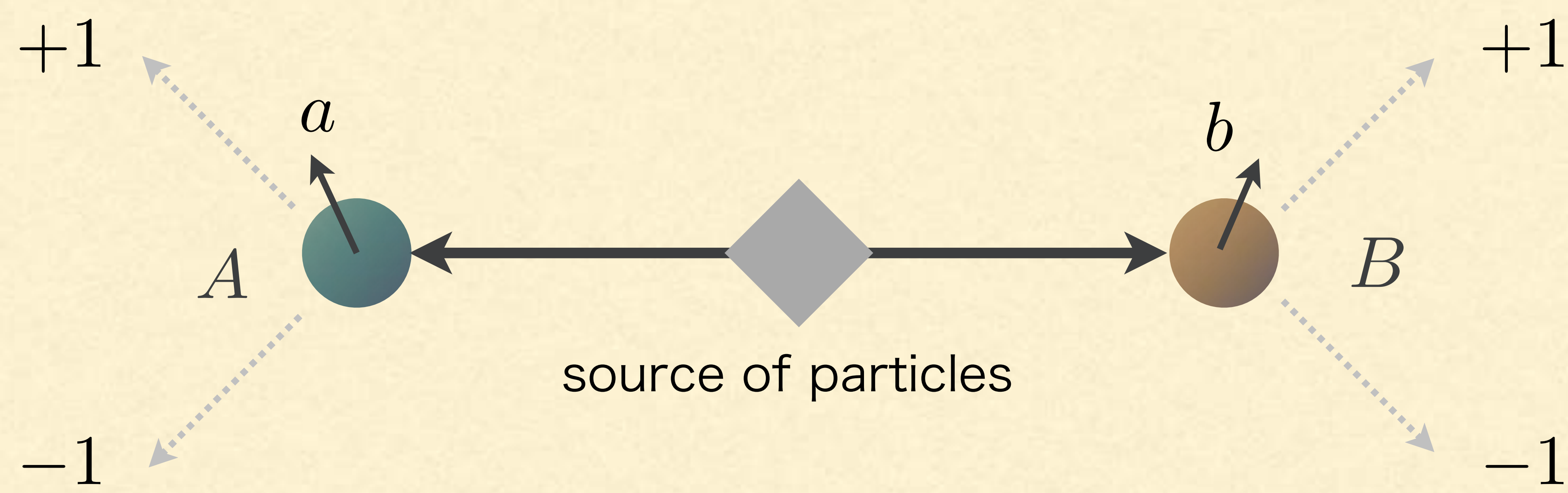
spin singlet case

$$S(\Delta\theta) \to |3\cos(\Delta\theta) - \cos(3\Delta\theta)|$$

QM breaks Bell inequality

contradiction between QM and local realistic theory

experimental test of Bell inequality



technical obstacles

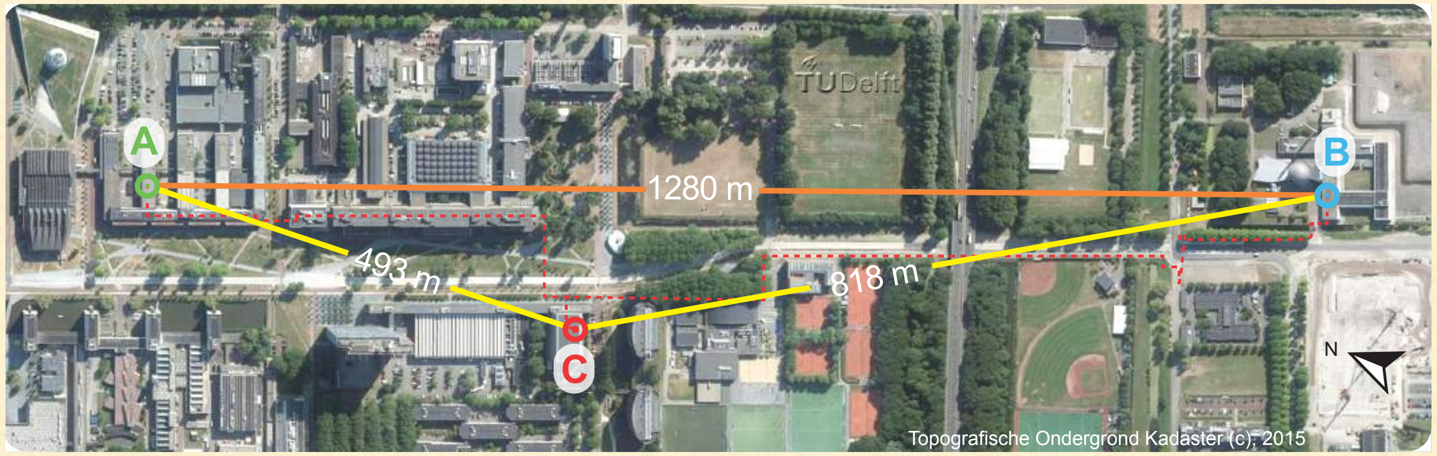
1) locality loophole — effect of measurement may be transmitted

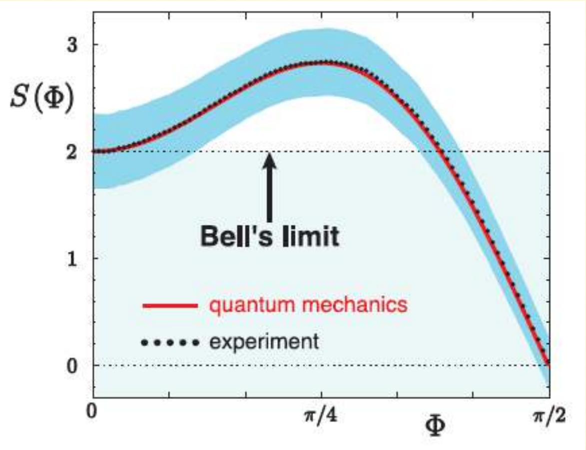
2) efficiency loophole — insufficient efficiency → fair sampling assumption

3) free choice loophole — choice of parameters may be predetermined

<u>recent tests & loopholes</u>

		locality	efficiency
Aspect et al. (1982)	photon: 12 m	△	×
Weihs et al. (1998)	photon: 400 m	○	×
Rowe et al. (2001)	ion	×	○
Sakai et al. (2006)	proton	△	△
Hensen et al. (2015)	electron	○	○
Giustina et al. (2015)	proton	○	○





4. Tools to quantify entanglement

For a system consisting two subsystems (bipartite system)

- Schmidt rank

$$|\Psi_{AB}\rangle = \sum_{i=0}^{d_A-1} \sum_{j=0}^{d_B-1} A_{ij}^{\Psi} |e_A^i\rangle \otimes |e_B^j\rangle$$

$$= \sum_{i=0}^{r(\Psi)} a_i |\tilde{e}_A^i\rangle \otimes |\tilde{e}_B^i\rangle$$ Schmidt decomposition

$r(\Psi) \leq \ min[d_A, d_B]$ If $r(\Psi) \neq 1$ → Entangled

- Von Nenmann entanglement entropy

$$\rho_A = \mathrm{Tr}_B|\Psi_{AB}\rangle\langle\Psi_{AB}|$$

$$\rho_B = \mathrm{Tr}_A|\Psi_{AB}\rangle\langle\Psi_{AB}|$$

Reduced density matrix

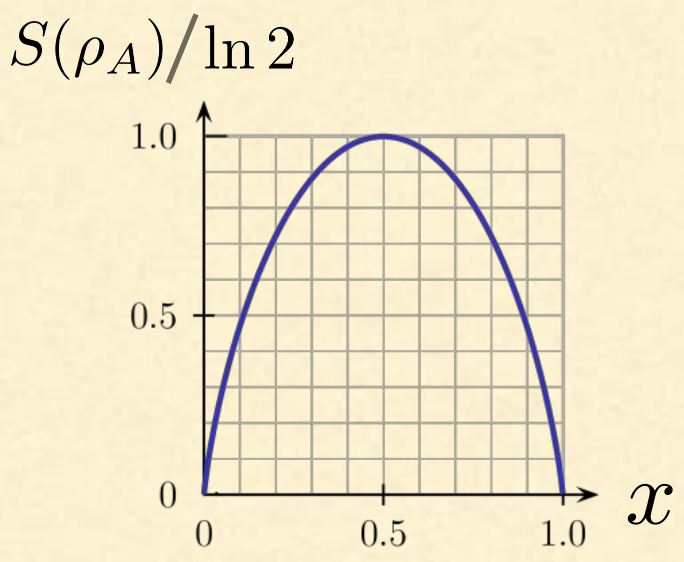
$$\mathcal{S}(\rho_A) = -\,\mathrm{Tr}[\rho_A \log \rho_A] = -\,\mathrm{Tr}[\rho_B \log \rho_B] = \mathcal{S}(\rho_B)$$

If $S(\rho_A) \neq 0$ → Entangled

ex)

$$|\Psi_{AB}\rangle = \alpha|\uparrow\rangle|\downarrow\rangle + \beta|\downarrow\rangle|\uparrow\rangle$$

$$|\alpha|^2 = x, \qquad |\beta|^2 = 1 - x$$



- Positive partial transpose (PPT) criterion

$$\rho_{AB} \overset{\text{PPT}}{\dashrightarrow} \rho_{AB}^{T_B} \quad \text{s.t.} \quad \langle m|\langle\mu|\varrho_{AB}^{T_B}|n\rangle|\nu\rangle \equiv \langle m|\langle\nu|\varrho_{AB}|n\rangle|\mu\rangle$$

ρ_{AB} Separable $\longleftrightarrow$ $\rho_{AB}^{T_B}$ density matrix

- Entangled witness $\quad \mathrm{Tr}(W\varrho_{AB}) \geq 0$
- Concurrence $\quad C(\psi) = \sqrt{1 - \mathrm{Tr}(\varrho_B^2)}$
- ...

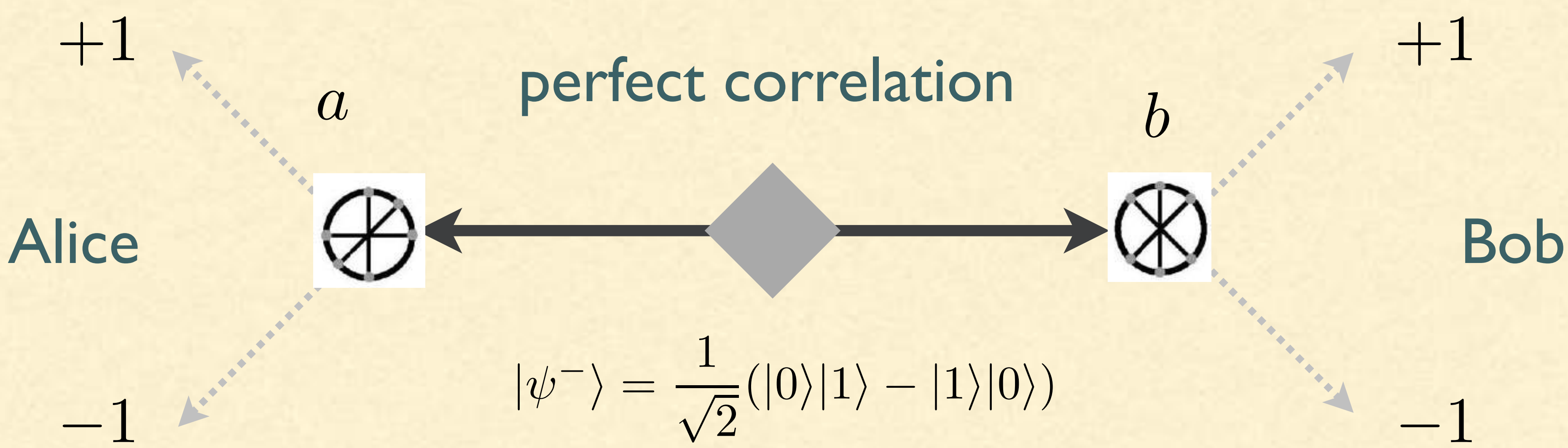
Remark: quantifying entanglement for multipartite mixed states is a difficult problem and still under investigation.

5. Some characteristic properties

Entanglement exhibits eminent properties to be used for various purposes

- Quantum key distribution

Ekert protocol



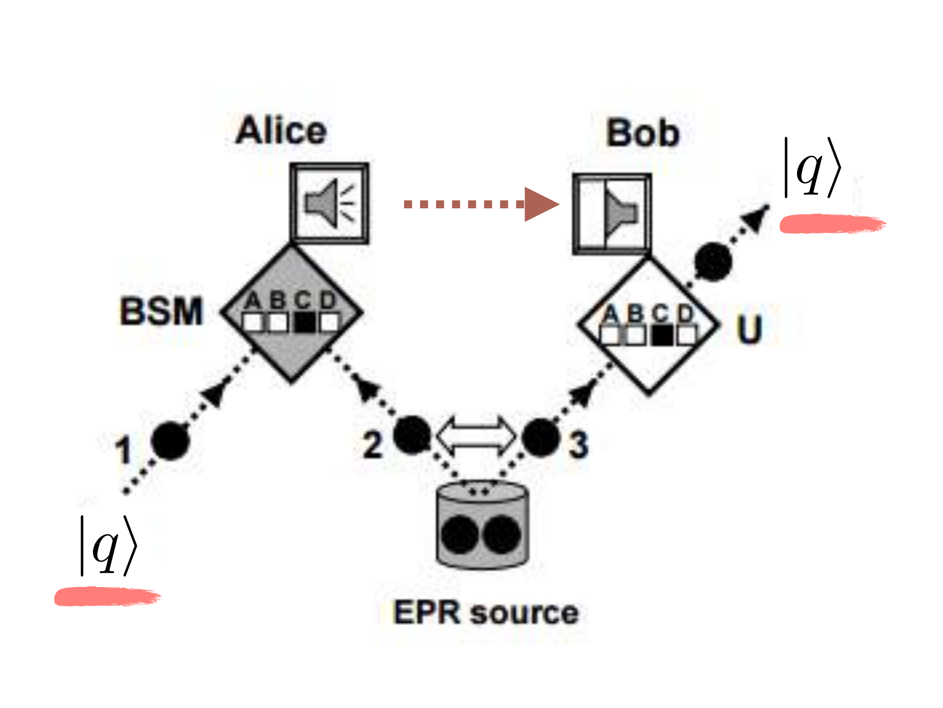
$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle - |1\rangle|0\rangle)$$

If someone (Eve) eavesdrops in between, then Bell's inequality is maintained

If not, Bell's inequality is broken, and the shared measured data can be used

- Quantum teleportation $|q\rangle = a|0\rangle + b|1\rangle$

$$|\psi_{AA'B}\rangle = |q\rangle_A \otimes \frac{1}{\sqrt{2}}[|0\rangle|0\rangle + |1\rangle|1\rangle]_{A'B}$$
$$= \frac{1}{2}[|\phi^+\rangle_{AA'}(a|0\rangle_B + b|1\rangle_B)$$
$$+|\phi^-\rangle_{AA'}(a|0\rangle_B - b|1\rangle_B)$$
$$+|\psi^+\rangle_{AA'}(a|1\rangle_B + b|0\rangle_B)$$
$$+|\psi^-\rangle_{AA'}(a|1\rangle_B - b|0\rangle_B)]$$



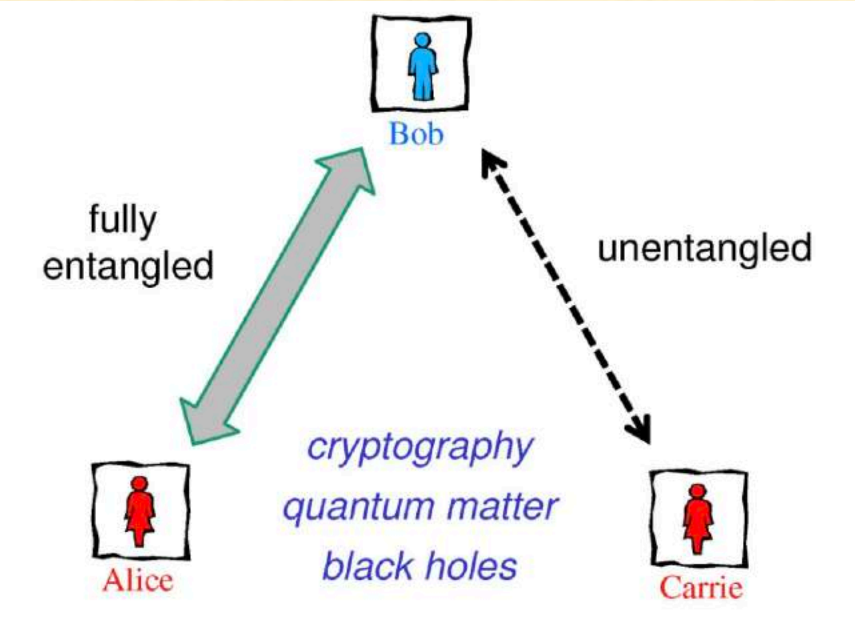
By measuring Alice's state with Bell basis, she can send her state to Bob

$$|q\rangle_A \longrightarrow |q\rangle_B$$

- Quantum computation
- …

- Monogamy of entanglement

If two qubits A and B are maximally correlated, they cannot be correlated at all with third qubit C



- Entanglement in long distance

Quantum entanglement records

Across the Danube	600 metres	2003 (Zeilinger group, Austria)
Great Wall of China	13 kilometres	2010 (Pan group, China)
Qinghai Lake	97 kilometres	2012 (Pan group)
Canary Islands	143 kilometres	2012 (Zeilinger group)
Micius satellite	1203 kilometres	2017 (Pan group)

Source: Science, Nature

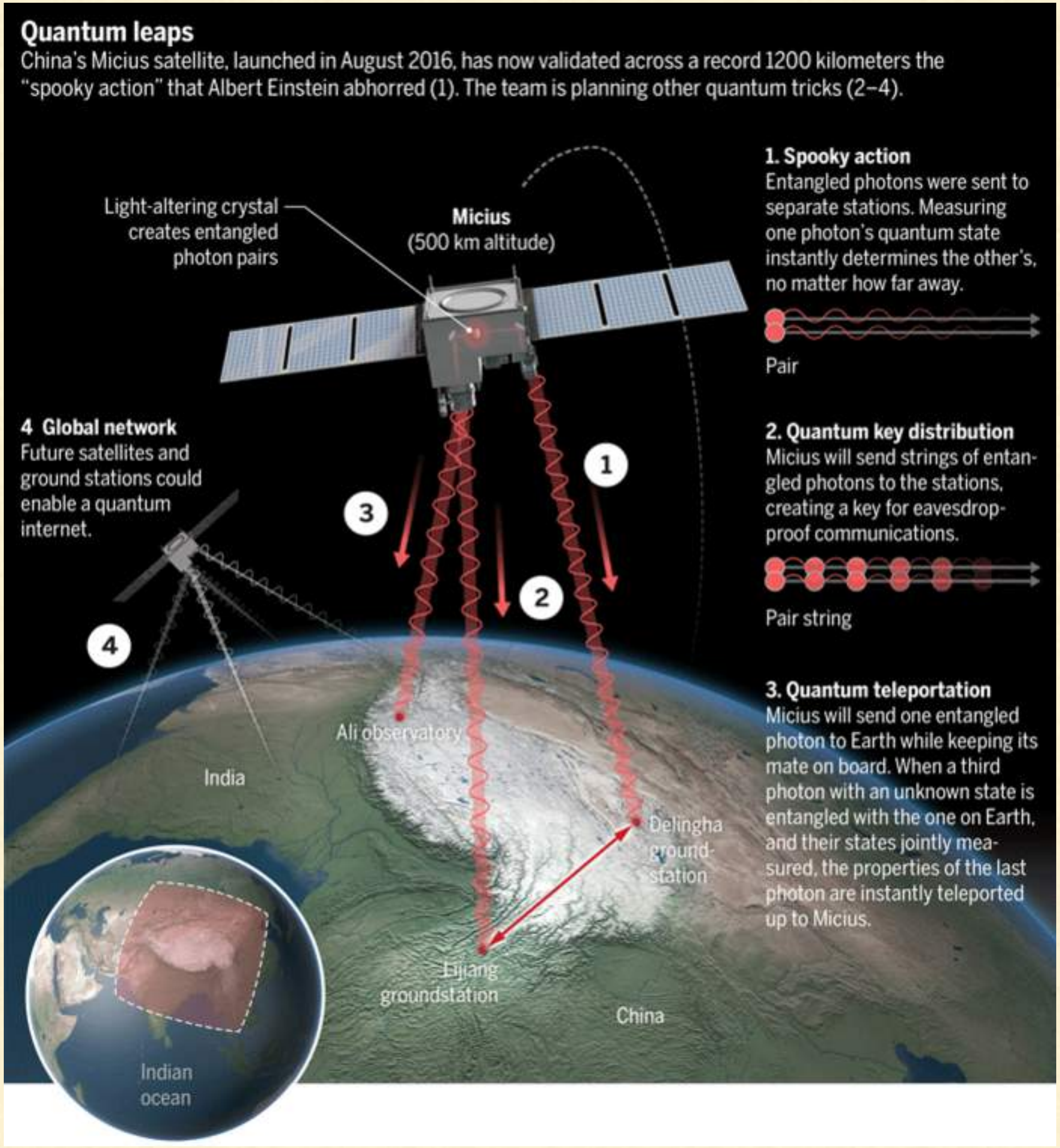
China’s satellite

Micius (墨子)

achieved entanglement at distance 1,203km (2017)

→

- quantum teleportation
- quantum key distribution
- global network (future)




Thank you!