On a New Approach to Continuum Path Integrals for Particles and Fields

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[1] T. S. , arXiv:1201.0055 [quant-ph]; full paper in preparation.

Introduction

++ Quantum physics ++

- **Quantum physics dominates various microscopic phenomena less than the atomic scale.**

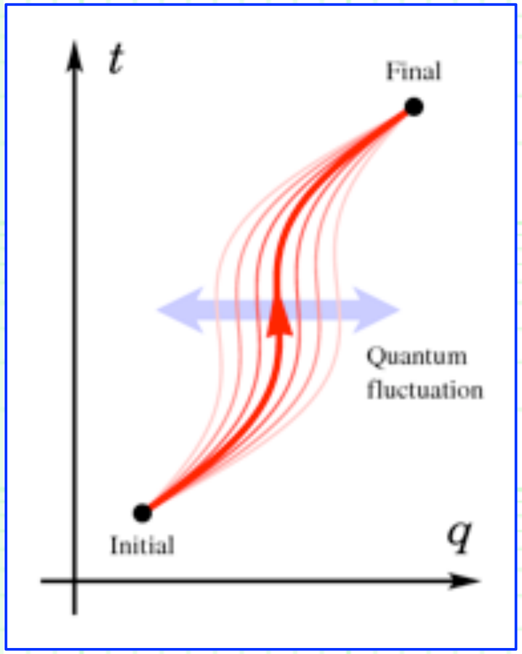
--> Quantization is the key to describe phenomena from viewpoint of microscopic dynamics.

--- The uses of quantum mechanics, quantum field theories, ...

- **Feynman's path integral: one elegant way to the quantization.**

$$\mathcal{Z} = \int \mathcal{D}q \exp(-S[q]), \quad \int \mathcal{D}q = \prod_{\tau} dq(\tau)$$

Feynman, *Rev. Mod. Phys.* 20 (1948) 367.



--- All possible paths are taken into account with the probability amplitude exp(-S) (S is action of the system).

--- Path integrals can also be used in quantum field theories as perturbative and non-perturbative techniques.

Introduction

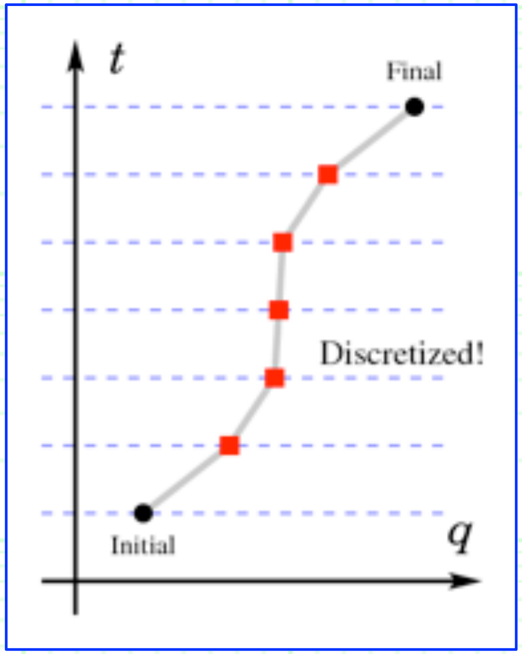
++ Time discretization for path integral ++

- **Path integrals can be simplified by discretizing time(-space).**
  - Derivation and integration --> finite difference and summation.
  - Easier to evaluate analytically and numerically.

- **Lattice QCD** is one of the most important example of the discretized path integrals.

  Wilson, *Phys. Rev.* D10 (1974) 2445.

--- Many non-perturbative aspects of QCD has been revealed from analytic and numerical discussions on lattice QCD.



- But **discretized theory is not continuum theory**.
  - Breaks time-space continuous symmetries down to discretized symmetries (e.g. translational symmetry).
  - Leads to qualitative discrepancies: **doubler** in the Dirac field, **magnetic monopoles** in lattice QED, ...

Wilson (1975). Polyakov, *Phys. Lett.* B59 (1975) 82.

Introduction

++ More physics from path integrals ++

- How can we extract, especially in quantum field theories, more (non-perturbative) quantum physics from path integrals ?

1. Make discretized approach close to the continuum theory.

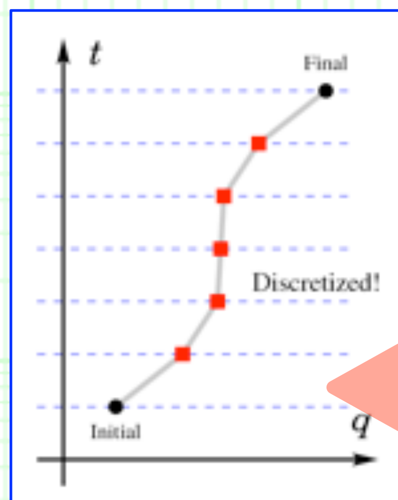
--- Small lattice spacing a, improved action, small quark mass, ...

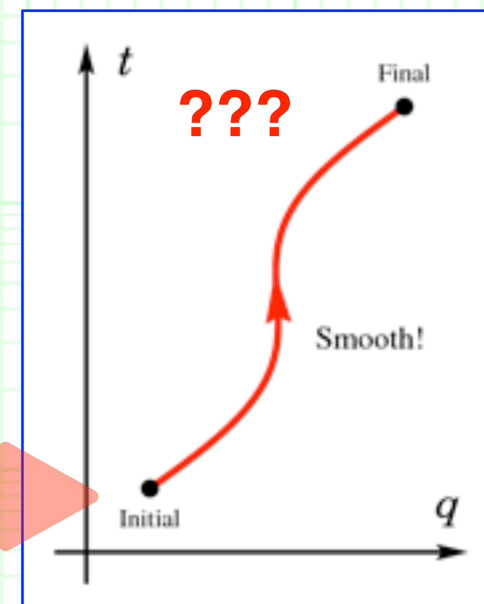
--> Precise determination of properties of QCD vacuum, excited states (hadrons), hadron interactions, ...

2. Simulation in continuum time-space without discretization???

--- Can we create (approximate) an approach to continuum path integrals?

--- It could be complementary (相補性) for the lattice simulations of quantum field theories.





Basic ideas

++ Discretized approach ++

- Review of discretized approach to simulations of path integrals.

Creutz and Freedman, *Ann. Phys.* 132 (1981) 427.

1. Consider additional fluctuation (change) of particle position at time j , which we denote δq_j , as a randomly-determined value:

$$\delta q_j \in [-\Delta, \Delta] \quad (\Delta\text{: a fixed value})$$

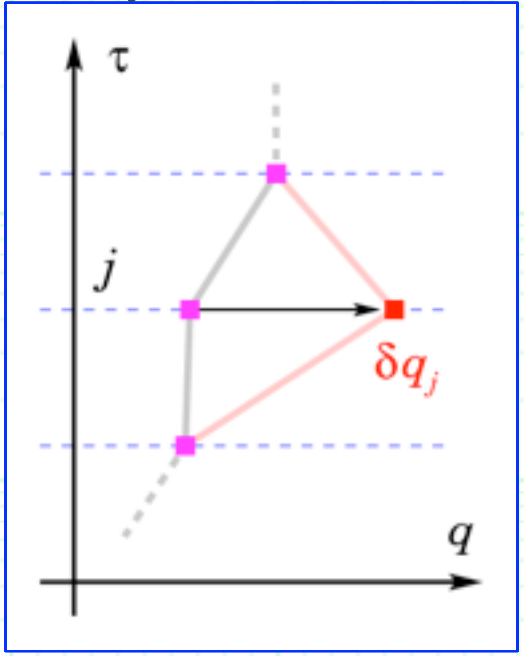
2. The additional fluctuation is judged by the Metropolis test: δq_j is accepted in probability

Metropolis et al., *J. Chem. Phys.* 21 (1953) 1087.

$$\min[1, \exp(S[q] - S[q + \delta q])]$$

Then if and only if δq_j is accepted, we redefine $q_j + \delta q_j$ as q_j. This gives weight exp(-S).

3. After several "sweeps", i.e., performing steps 1 and 2 from j=1 to N_{lat}, quantum paths in equilibrium with weight exp(-S) are obtained.



Basic ideas

++ Discretized approach ++

- Review of discretized approach to simulations of path integrals.

Creutz and Freedman, *Ann. Phys.* 132 (1981) 427.

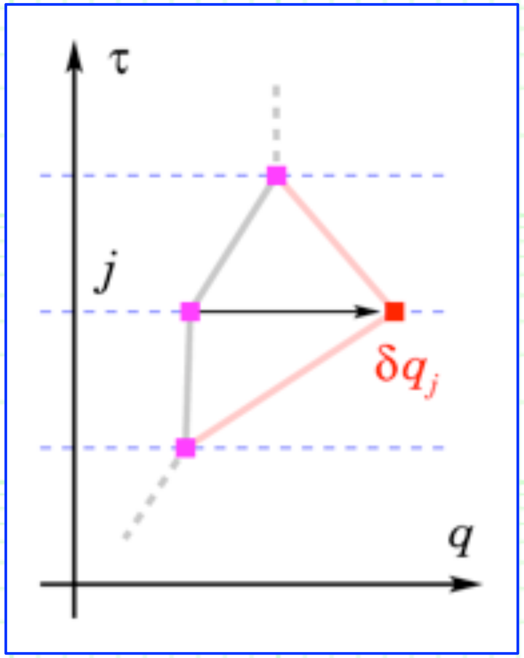
--> 3 lessons from the discretized approach:

i) Every time j is equally treated without making any special time.

ii) Micro-reversibility for additional fluctuation δq_j : without making any specific directions.

iii) δq_j is judged by the Metropolis method (or others) to give weight exp(-S) to the paths.

- Indeed, with above 3 points we can create a procedure for the discretized path integrals which leads to quantum paths in equilibrium.



Basic ideas

++ Our approach to continuum theory ++

- We develop **our approach to the continuum path integrals**.

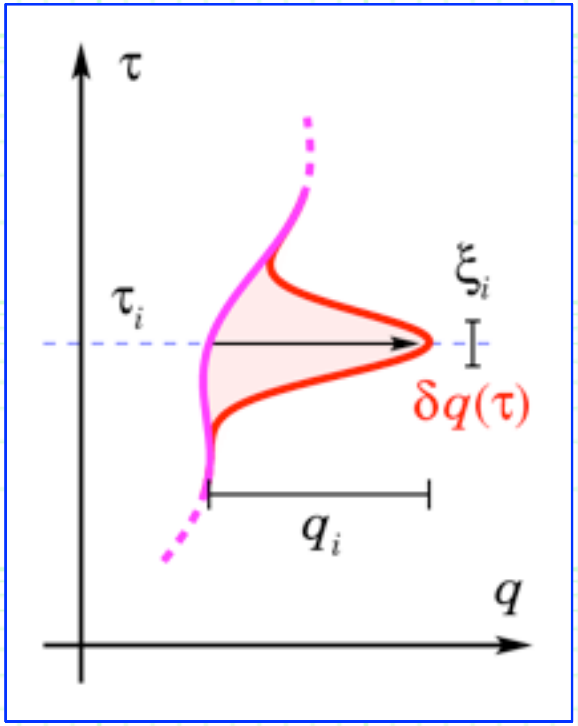
--- Describe paths by sum of Gauss functions with weight exp(-S).

$$q(\tau) = \sum_i q_i \exp\left[-\frac{(\tau-\tau_i)^2}{\xi_i^2}\right]$$

<-- **Fluctuations approximated by the Gauss functions.**

- We need the constant set (q_i, τ_i, ξ_i):
  - q_i: amplitude of each fluctuation.
  - τ_i: time component of each fluctuation.
  - ξ_i: width (scale) of each fluctuation.
    (corresponds to the lattice spacing a)

--- **Determine so as to give weight exp(-S).**



Basic ideas

++ Our approach to continuum theory ++

- **Lessons from discretized approach:**

i) Every time is equally treated. ii) Micro-reversible δq_j.

iii) Metropolis test for weight exp(-S).

- **Our approach to construct smooth path:**

$$q(\tau) = \sum_i q_i \exp\left[-\frac{(\tau-\tau_i)^2}{\xi_i^2}\right]$$

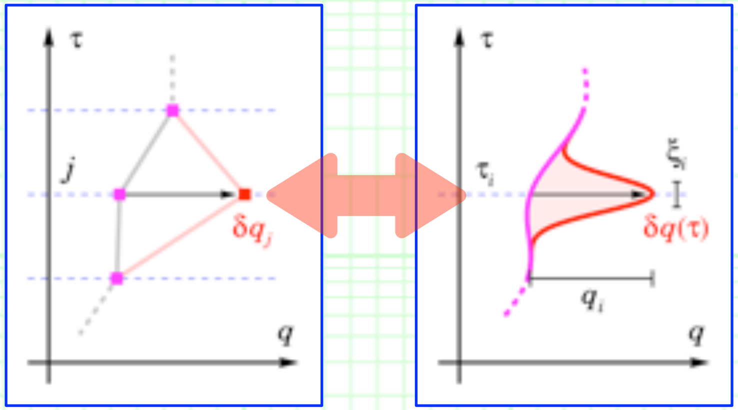
$$\delta q(\tau) = q_i \exp\left[-\frac{(\tau-\tau_i)^2}{\xi_i^2}\right]$$

1. Determine initial smooth path.
2. Construct additional fluctuation $\delta q(\tau)$ with (q_i, τ_i, ξ_i) generation:

$$\begin{cases} q_i \in [-\Lambda_q, \Lambda_q], & \text{generated in uniform probability} \\ \tau_i \in [0, \mathcal{T}], & \text{generated in uniform probability} \\ \xi_i: & \text{fixed in this study} \end{cases}$$

--- Uniformity is key to lessons i) and ii).

Especially $\delta q(\tau)$ is micro-reversible!



Basic ideas

++ Our approach to continuum theory ++

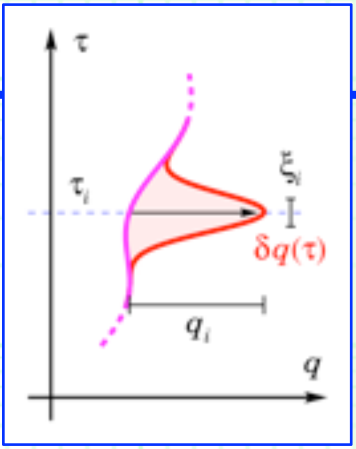
- **Lessons from discretized approach**:

  i) Every time is equally treated. ii) Micro-reversible δq_j.

  iii) Metropolis test for weight exp(-S).

- **Our approach to construct smooth path**:

  3. Additional fluctuation $\delta q(\tau)$



$$\delta q(\tau) = q_i \exp\left[-\frac{(\tau - \tau_i)^2}{\xi_i^2}\right]$$

is judged by the Metropolis test. If and only if $\delta q(\tau)$ is accepted, we redefine $q(\tau) + \delta q(\tau)$ as $q(\tau)$.

--- Key to lesson iii).

4. Iterate steps 2 and 3 until the action (and others) converges.

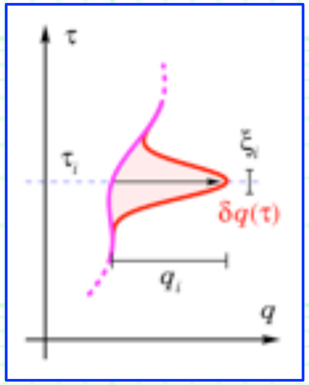
--> Eventually we obtain smooth path as sum of $\delta q(\tau)$, i.e. in the following form:

$$q(\tau) = \sum_i q_i \exp\left[-\frac{(\tau - \tau_i)^2}{\xi_i^2}\right]$$

Basic ideas

++ Our approach to continuum theory ++

- **Our approach to construct smooth paths is summarized as:**
  1. Determine initial smooth path.
  2. Construct additional fluctuation $\delta q(\tau)$: $$\delta q(\tau) = q_i \exp\left[-\frac{(\tau-\tau_i)^2}{\xi_i^2}\right]$$
  3. Additional fluctuation $\delta q(\tau)$ is judged by the Metropolis test. If and only if $\delta q(\tau)$ is accepted, we redefine $q(\tau) + \delta q(\tau)$ as $q(\tau)$.
  4. Iterate steps 2 and 3 until the action (and others) converges.



--> **Obtain smooth path**: $$q(\tau) = \sum_i q_i \exp\left[-\frac{(\tau-\tau_i)^2}{\xi_i^2}\right]$$

- Expression of paths (fluctuations) only by certain function (the Gauss function in this study) is an "approximation".

--> We give weight exp(-S) to paths of such an expression.

--- How good is this expression?

Results for particles

++ Harmonic oscillator in d=1 ++

- Let us examine our approach with harmonic oscillator in d=1:

$$S_{\rm HO} = \int_0^{\mathcal{T}} d\tau L_{\rm HO}(q,\dot{q}), \quad L_{\rm HO}(q,\dot{q}) = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}m\omega^2 q^2$$

with our smooth paths:

$$q(\tau) = \sum_i q_i \exp\left[-\frac{(\tau-\tau_i)^2}{\xi_i^2}\right]$$

- Here we fix m=ω=1, and take periodic boundary condition with: Time range T=200, fluctuation cut-off Λ_q=3, and width ξ=1.3. (ξ is fixed as the peak position of ξ-histogram in random ξ case).

- We take a "hot start" by randomly generating 400 (q_i, τ_i, ξ_i) sets, and prepare N=100 paths for the statistical treatment.

- Temperature of system =1/T=1/200 << ω=1.

--> Our paths will reflect ground state of the harmonic oscillator:

$$\psi_{\rm GS}(q) = \left(\frac{1}{\pi}\right)^{1/4} \exp\left(-\frac{q^2}{2}\right)$$

-->

$$\langle q^2\rangle_{\rm GS} = 0.5, \quad \left\langle K\left(\equiv \frac{1}{2}\dot{q}^2\right)\right\rangle_{\rm GS} = \left\langle V\left(\equiv \frac{1}{2}q^2\right)\right\rangle_{\rm GS} = 0.25$$

Results for particles

++ Harmonic oscillator in d=1 ++

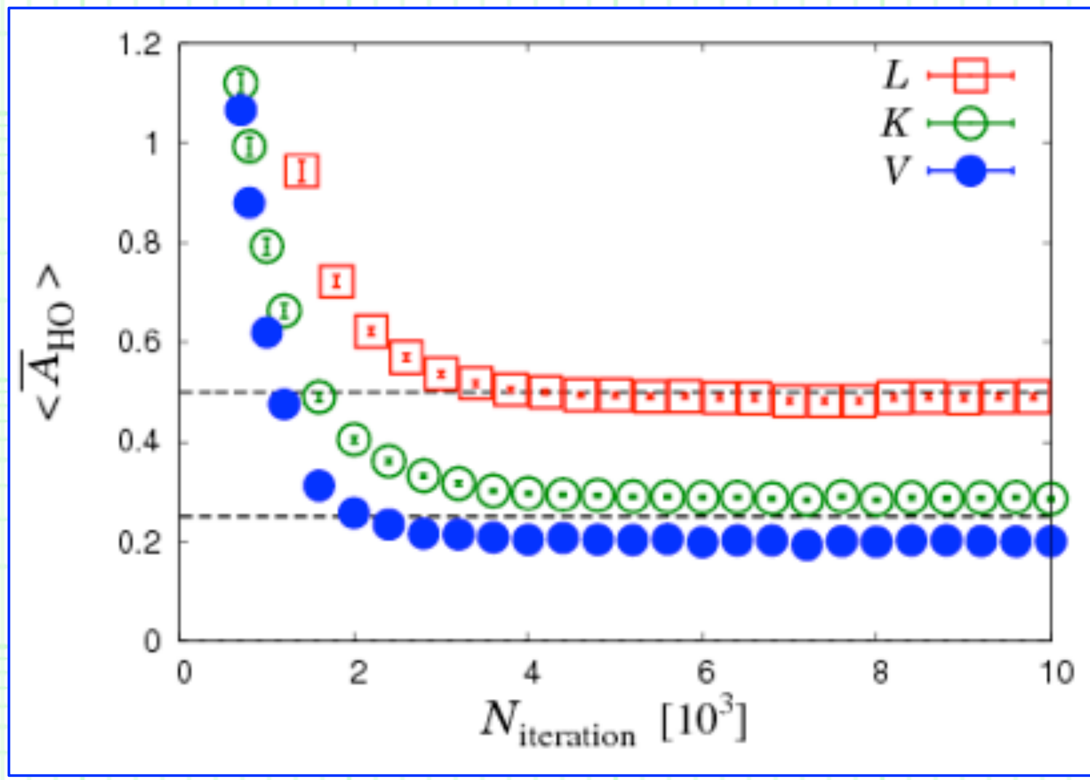
- **Cooling behavior of harmonic oscillator in d=1 with our smooth paths:**

$$q(\tau) = \sum_i q_i \exp\left[-\frac{(\tau-\tau_i)^2}{\xi_i^2}\right]$$

- **Lagrangian, kinetic, and potential expectation values converge around $N_{\mathrm{iteration}}$ ~ 6 x 10^3 with ~ 2000 Gauss function.**

- **After $N_{\mathrm{iteration}} = 10^4$, we have**
  $\langle K \rangle = 0.286 \pm 0.003$,
  $\langle V \rangle = 0.200 \pm 0.002$.

<--> $\langle K\rangle_{\mathrm{GS}} = \langle V\rangle_{\mathrm{GS}} = 0.25$



- **We can reproduce quantum values of harmonic oscillator with 80-90% accuracy.**

Results for particles

++ Harmonic oscillator in d=1 ++

- **Example of the quantum fluctuation** (right figure, up to τ=50).

<-- Expression:

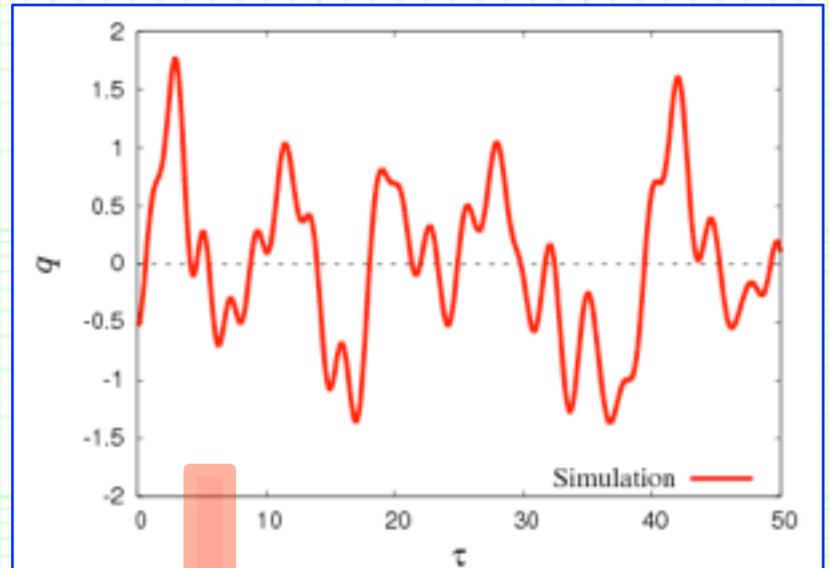
$$q(\tau) = \sum_i q_i \exp\left[-\frac{(\tau-\tau_i)^2}{\xi_i^2}\right]$$

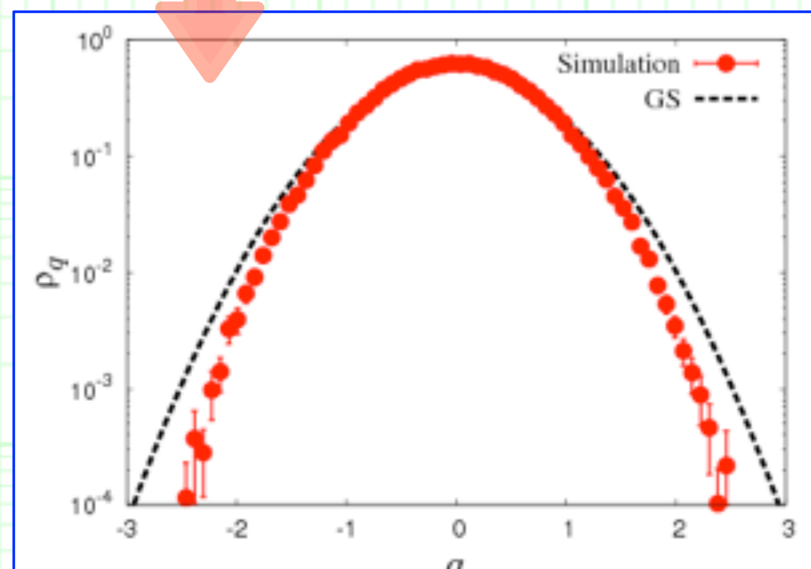
--- $q(\tau)$ fluctuates from zero to ~ ± 2, but within $|q| \lesssim 1$ at most times.
($\langle q^2 \rangle = 0.400 \pm 0.005$)

--- Peak structures with width $\lesssim 3$.

- **Visualize degree of quantum fluctuation as q-distribution.**

--> Our q-distribution behaves similarly compared to the squared ψ_{GS}.



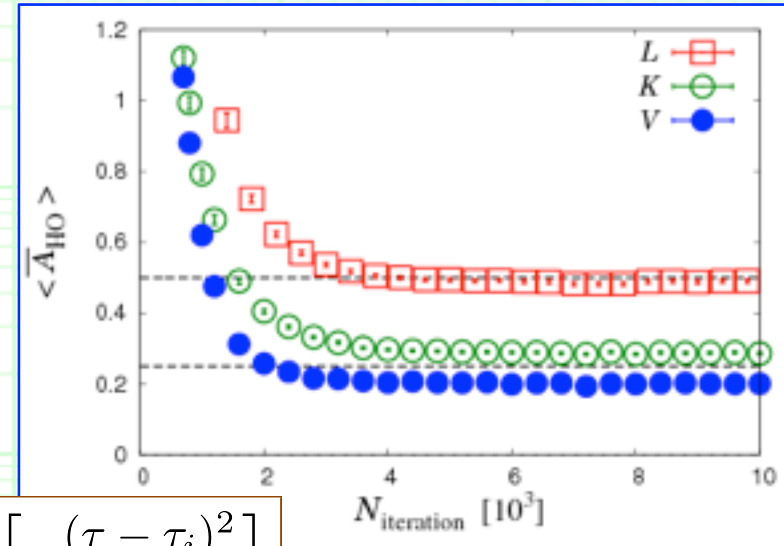


Results for particles

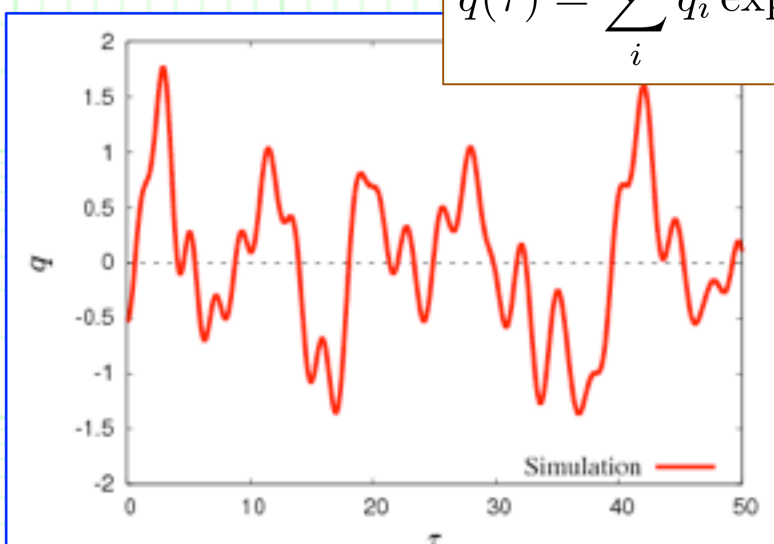
++ Summary for harmonic oscillator ++

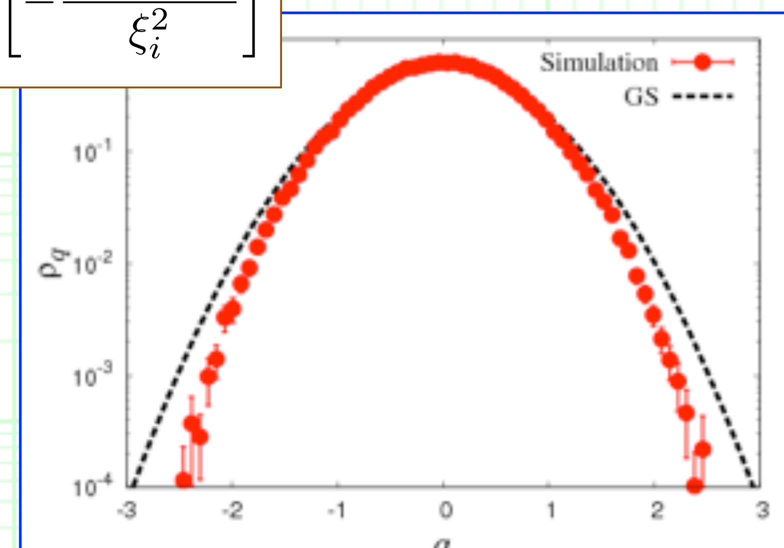
- Discussions on harmonic oscillator indicate that **our approach qualitatively reproduces quantum behaviors of the system with 80-90% accuracy**, even though **paths are expressed only by the Gauss function**.

--- **Our approach is approximate evaluation, but gives qualitative results.**



$$q(\tau) = \sum_i q_i \exp\left[-\frac{(\tau-\tau_i)^2}{\xi_i^2}\right]$$



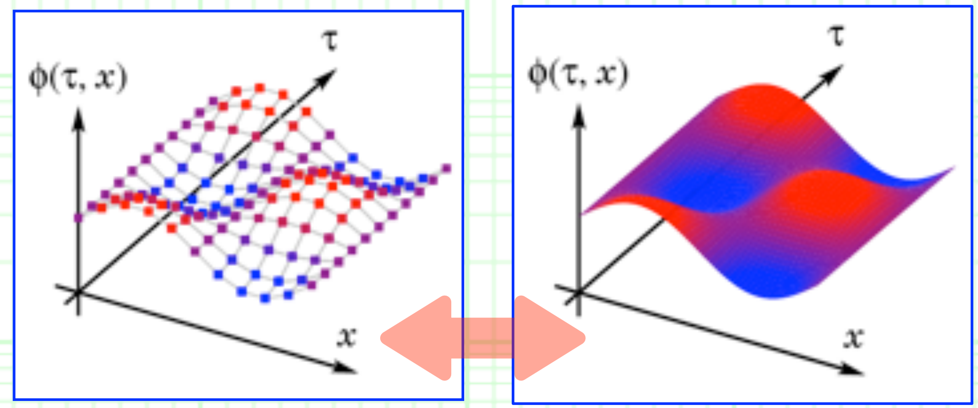


Results for fields

++ Quantum fluctuations for fields ++

- **We can apply our approach of the continuum path integrals also to quantum fluctuations for fields.**

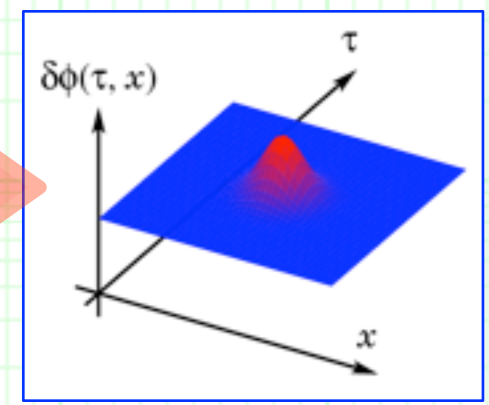
$$\mathcal{Z} = \int \mathcal{D}\phi \exp(-S[\phi]), \quad \mathcal{D}\phi \equiv \prod_x d\phi(x)$$



- **Additional fluctuations for fields rather than particles:**

--- The only difference is number of components for x_i.

$$\delta\phi(x) = \phi_i \exp\left[-\frac{(x-x_i)^2}{\xi_i^2}\right]$$



- **Here we assume periodic boundary condition for finite 4D box, (X, X, X, T).**

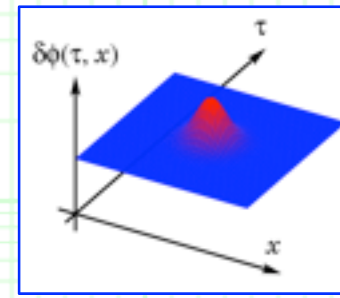
Results for fields

++ Our approach to continuum theory ++

- **Our approach to construct smooth "path" for fields:**
  1. Determine initial smooth path.
  2. Construct additional fluctuation $\delta\phi(x)$:

$$\delta\phi(x) = \phi_i \exp\left[-\frac{(x-x_i)^2}{\xi_i^2}\right]$$

     Constant set (ϕ_i, x_i, ξ_i) is generated as:



$$\begin{cases} \phi_i \in [-\Lambda_\phi, \Lambda_\phi], & \text{generated in uniform probability} \\ x_i \in (\mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{T}), & \text{generated in uniform probability} \\ \xi_i\text{: fixed in this study} & \text{—corresponds to the minimal scale} \end{cases}$$

  3. Additional fluctuation $\delta\phi(x)$ is judged by the Metropolis test.
     If and only if $\delta\phi(x)$ is accepted, we redefine $\phi(x) + \delta\phi(x)$ as $\phi(x)$.
  4. Iterate steps 2 and 3 until the action (and others) converges.

--> **Obtain smooth "path":**

$$\phi(x) = \sum_i \phi_i \exp\left[-\frac{(x-x_i)^2}{\xi^2}\right]$$

- Again, expression of fluctuations only by certain function (the Gauss function in this study) is an "approximation".

Results for fields

++ Our approach to continuum theory ++

- **Our approach to construct smooth "path" for fields:**

$$\phi(x) = \sum_i \phi_i \exp\left[-\frac{(x-x_i)^2}{\xi^2}\right]$$

- **However, there is one obstacle for fields:**
  we will need ~ 10^{13} Gauss functions for fields in d=4.
  (cf. ~ 2000 Gauss functions for particles in d=1).

--- Hard to numerically calculate with ~ 10^{13} Gauss functions.

--> We restrict x_i on sites of 4D lattice dividing the 4D box in same intervals so that number of the Gauss functions unchanged.

--- Another approximation in our approach.

--- cf. randomly-chosen points in the 4D box for x_i.

Results for fields

++ U(1) and SU(2) gauge theories ++

- **Apply our approach to U(1) and SU(2) gauge theories in d=4:**

$$S = \int d^4x \mathcal{L}(x), \quad \mathcal{L}_{U(1)} = \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2, \quad \mathcal{L}_{SU(2)} = \frac{1}{4}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon_{abc}A_\mu^b A_\nu^c\right)^2$$

with our smooth paths: $A_\mu^{(a)}(x) = \sum_{i_{\mu(a)}} A_{i_{\mu(a)}} \exp\left[-\frac{(x - x_{i_{\mu(a)}})}{\xi^2}\right]$

--- In this study we do not include gauge fixing terms nor ghosts.
--> Our approach takes into account contributions from all of the gauge copies within the fluctuation cut-off Λ_A.

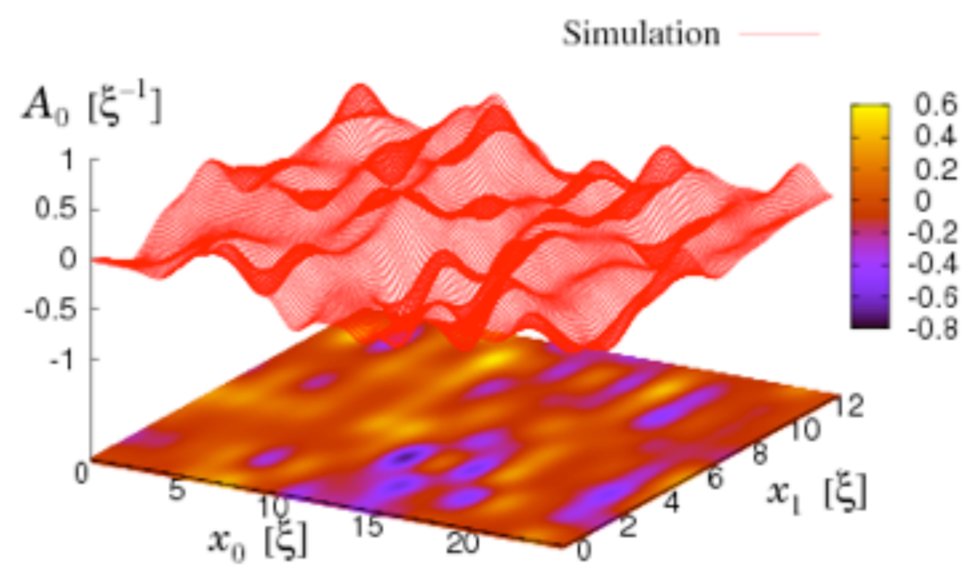
- **Here we take conditions:**
  4D box size T=2X with x_i-site lattice (N_x, N_t)=(7, 14), fluctuation cut-off Λ_A=1.3 ξ^{-1}, and scale $\xi = X/(N_x\sqrt{\pi})$.

- **We take a "hot start" by randomly generating A_i at every site x_i and prepare N=50 paths for the statistical treatment.**

Results for fields

++ Quantum fluctuation of U(1) gauge field ++

- **Example of quantum fluctuation of U(1) gauge field** (A_0 at certain x_2 and x_3)



- **We can calculate potential between fundamental representation and its anti-particle via expectation value of the Wilson loop,**

  Wilson, *Phys. Rev.* D10 (1974) 2445.

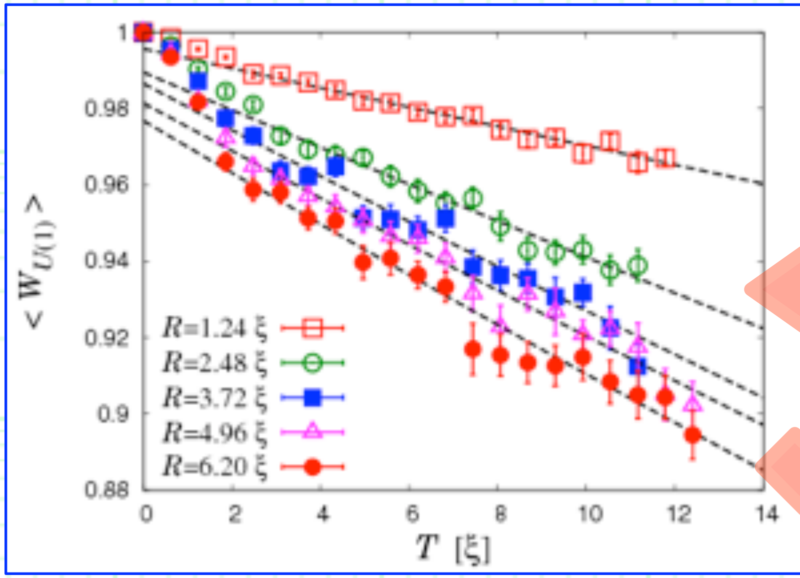
$$W_{U(1)}(C) = \exp\left[ie \oint_C dx_\mu A_\mu(x)\right]$$

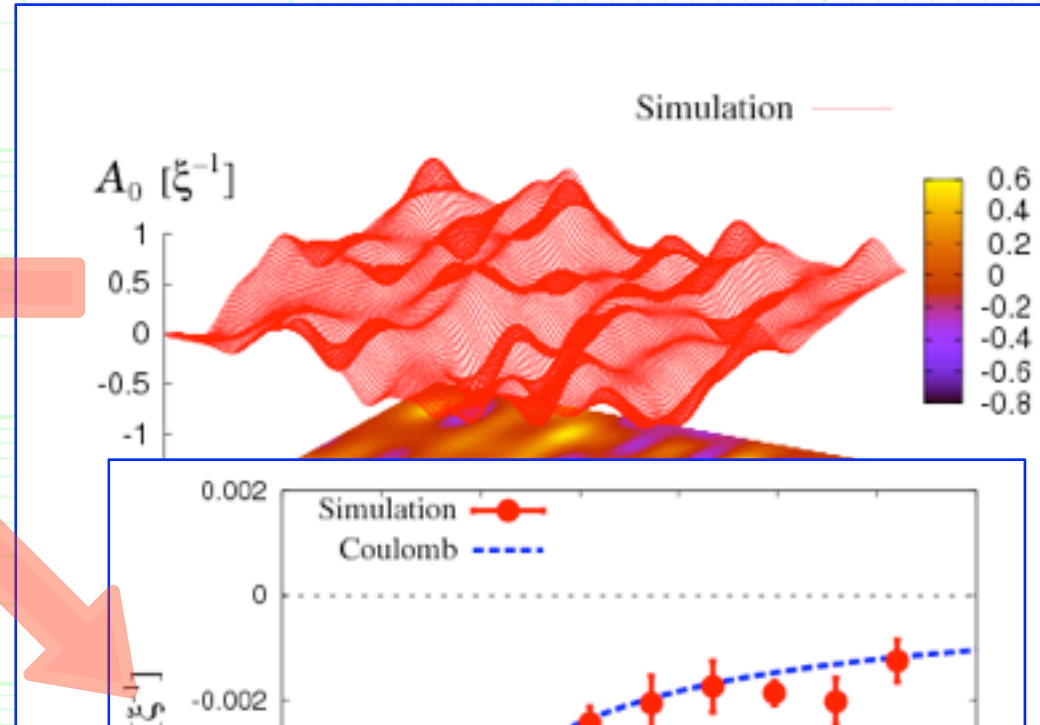
- **Here we use electric charge e=0.303 for U(1) so that α~1/137. Furthermore, we calculate average of 10 Wilson loops in each path for enough statistics.**

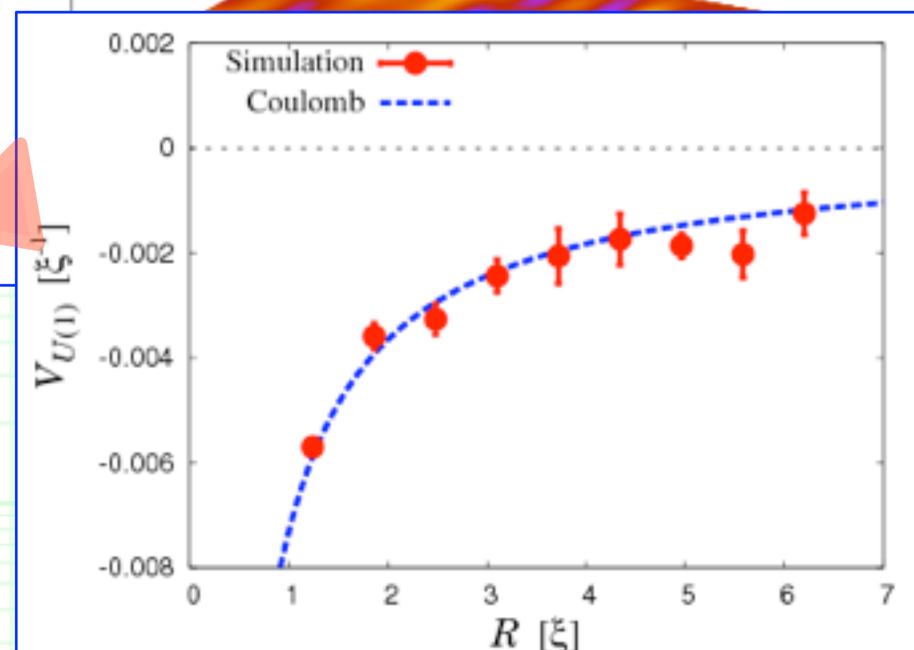
Results for fields

++ Quantum fluctuation of U(1) gauge field ++

- **Example of quantum fluctuation of U(1) gauge field**







- From the Wilson loop we can **reproduce the Coulomb force**.

--- Not so clear results.

<-- Intrinsic qualitativeness of our approach (“approximations”) + not enough statistics.

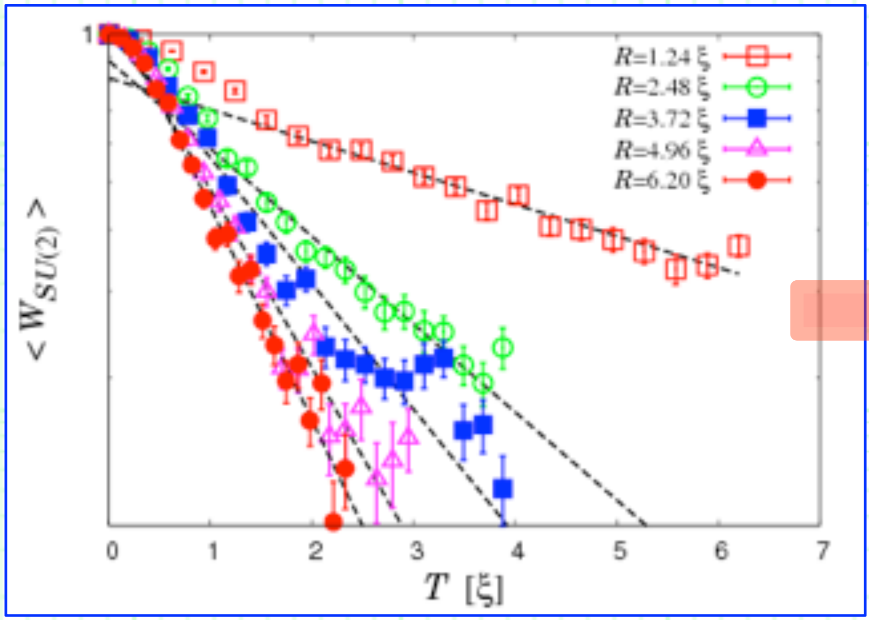
Results for fields

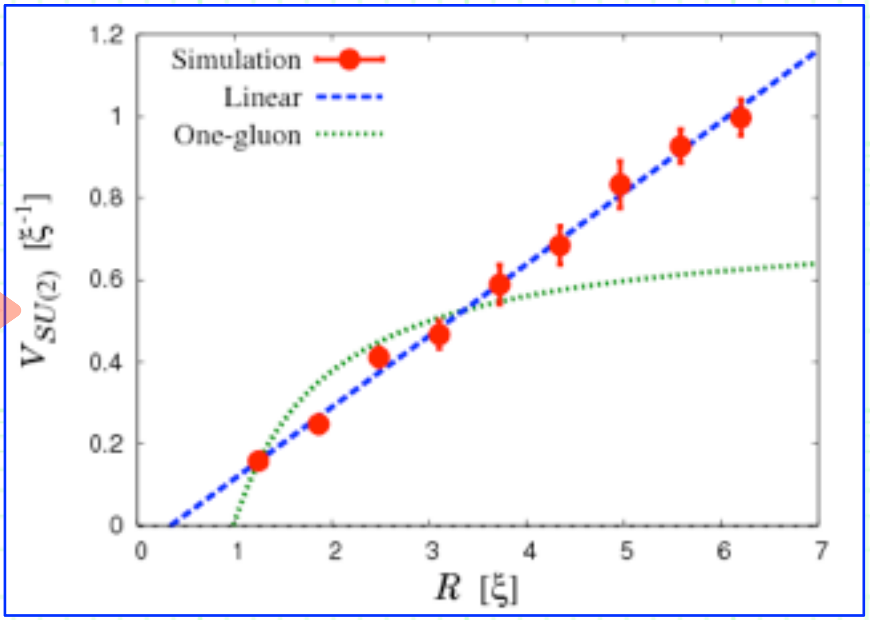
++ Quantum fluctuation of SU(2) gauge field ++

- **In a similar way calculate SU(2) potential from the Wilson loop:**

[coupling g=3.5 for SU(2)]

$$W_{SU(N_c)}(C) = \frac{1}{N_c}\mathrm{tr}\mathcal{P}\exp\left[ig\oint_C dx_\mu A^a_\mu(x)T^a\right]$$



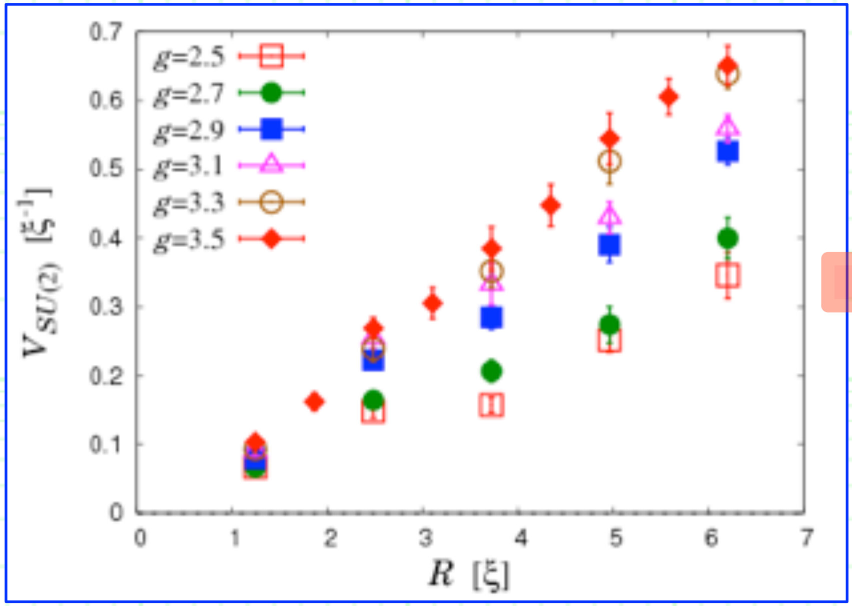


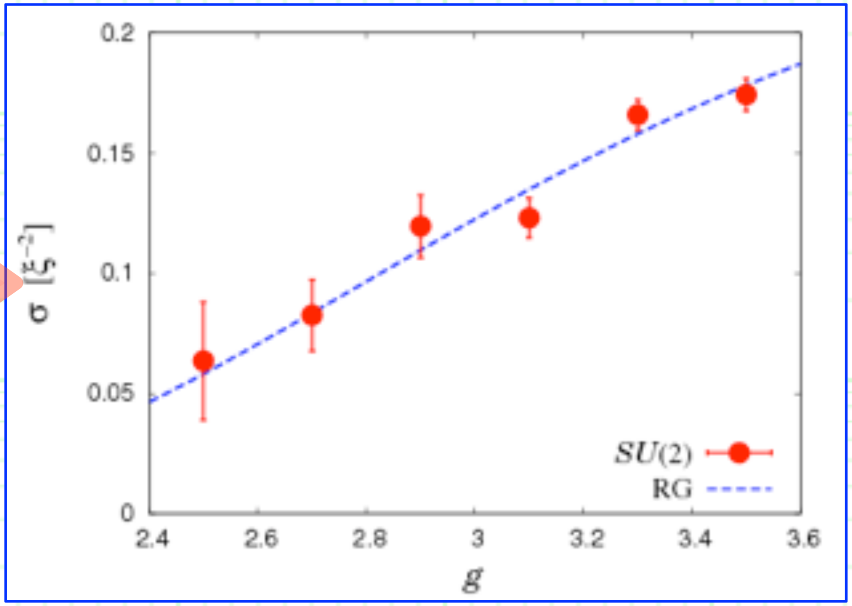
--- SU(2) potential in our approach deviates from the one-gluon exchange potential, but shows confining linear potential !

Results for fields

++ Quantum fluctuation of SU(2) gauge field ++

- Behavior of the SU(2) string tension with respect to coupling g:





--- Consistent with the renormalization group prediction:

$$\sigma_{SU(2)} \times \xi^2 \propto \left(\frac{24\pi^2}{11g^2}\right)^{102/121} \exp\left(-\frac{24\pi^2}{11g^2}\right).$$

--> Supports that our approach approximately describes the ground state of SU(2) gauge theory.

Summary

++ Summary ++

- We have developed an approach to the simulations of the continuum path integrals.

--- Paths (fluctuations) are described by sum of smooth functions with weight exp(-S) by the Metropolis method.

<-- Expression by certain function is an approximation.

- We obtain qualitative results with 80-90% accuracy on quantum behaviors of harmonic oscillator in d=1.

- We have evaluated quantum fluctuations of fields, fixing the fluctuation position x_i --- another approximation.

---> The Coulomb force and confining linear potential are extracted from the U(1) and SU(2) gauge fields in d=4 via Wilson loops, respectively, at qualitative levels.

--- Behavior of the SU(2) string tension is consistent with the renormalization group prediction.

Summary

++ Outlook ++

- More precise calculations for quantum systems.

--- By using "better" smooth functions such as "complete set", ...

- Our results are at qualitative levels.

--- Intrinsic qualitativeness of our approach (with Gauss functions)
 + not enough statistics (for quantum field theory).

--> Our approach could be complementary (相補性) for the lattice simulations:

Lattice -- U_μ vs. Our approach -- A_μ.

--> Our approach might be useful to investigate dynamics for non-perturbative fields such as confinement, spontaneous chiral symmetry breaking.

(cf. Instantons and monopoles can be directly described in terms of gauge fields A_μ rather than link variable U_μ.)

Thank you very much for your kind attention!

Appendix

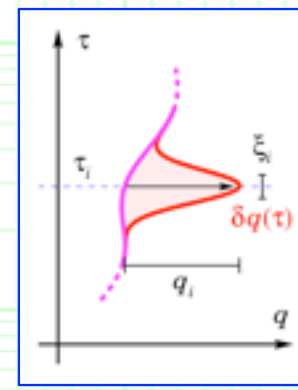
Appendix

++ Our approach to continuum theory ++

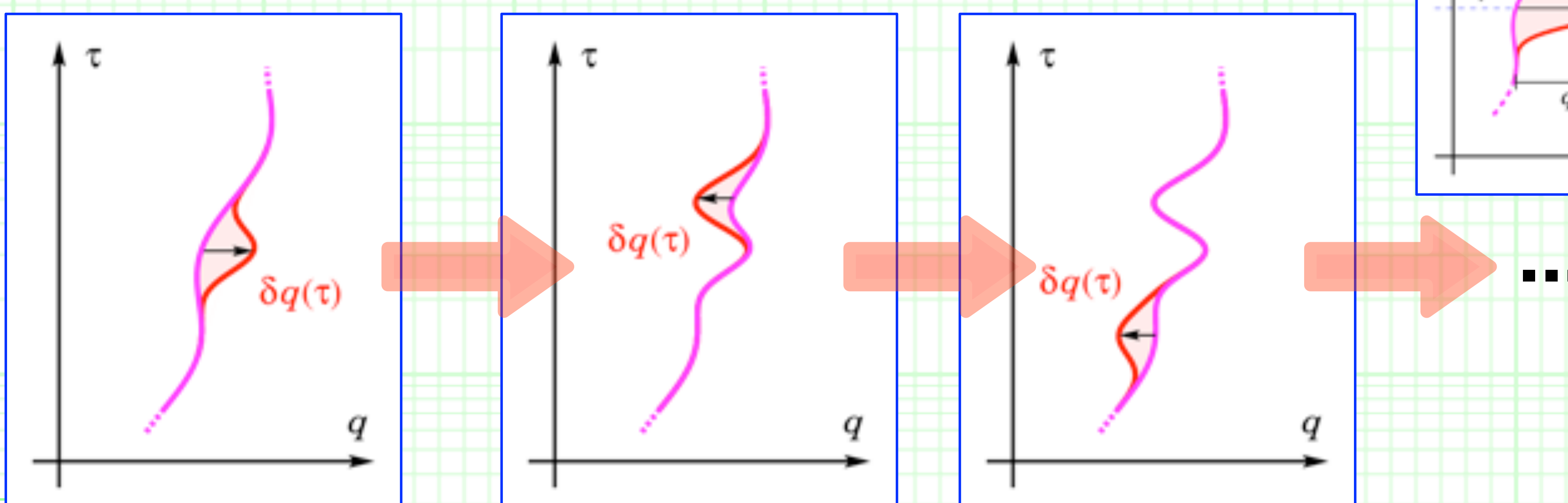
- Our approach to construct smooth path:
  1. Determine initial smooth path.
  2. Construct <u>additional fluctuation $\delta q(\tau)$</u>:

$$\delta q(\tau) = q_i \exp\left[-\frac{(\tau - \tau_i)^2}{\xi_i^2} \right]$$

  3. Additional fluctuation $\delta q(\tau)$ is <u>judged by the Metropolis test</u>.
     If and only if $\delta q(\tau)$ is <u>accepted</u>, we <u>redefine $q(\tau) + \delta q(\tau)$ as $q(\tau)$</u>.
  4. <u>Iterate steps 2 and 3</u> until the action (and others) converges.



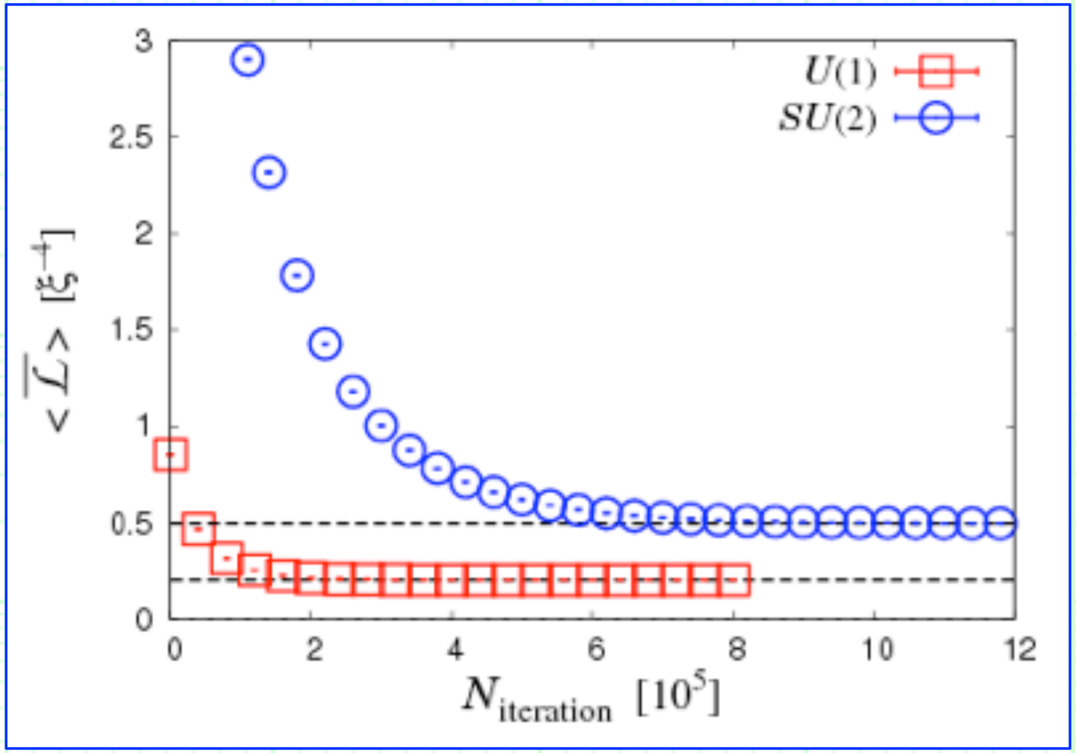
ex.)



Appendix

++ U(1) and SU(2) gauge theories ++

- **Cooling behaviors show that the Lagrangian densities converge:** $N_{iteration} \sim 3 \times 10^5$ **for U(1) and** $N_{iteration} \sim 10^6$ **for SU(2)**
  [coupling g=3.5 for SU(2)]
- **It is interesting that** $< L >_{SU(2)} \approx 0.49\ \xi^{-4}$ **is smaller than** $3 \times < L >_{U(1)} \approx 3 \times 0.20\ \xi^{-4}$ **due to the self-interactions in SU(2).**



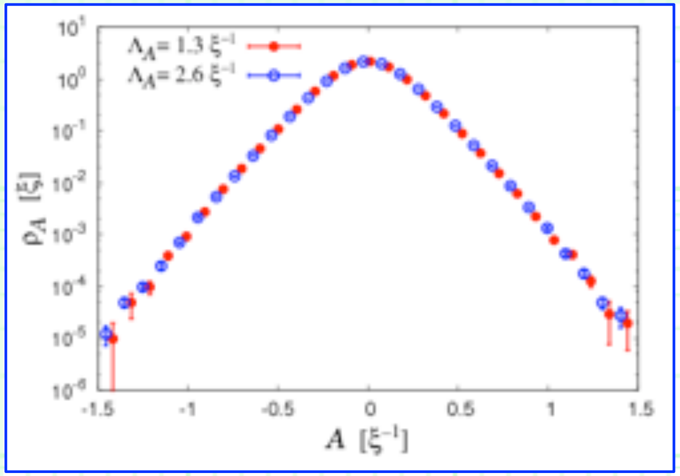
Appendix

++ Gauge symmetry in our approach ++

- We do not include gauge fixing terms nor ghosts in Lagrangian.

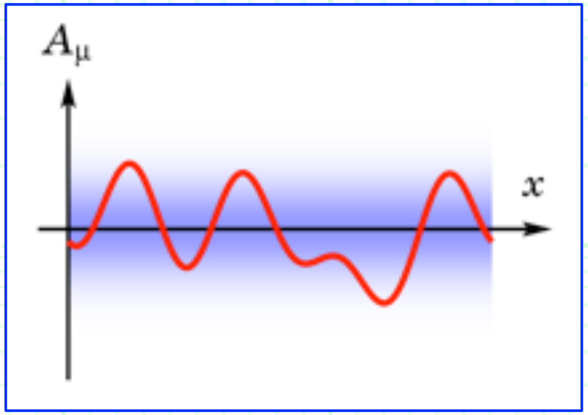
--> Our approach takes into account contributions from all of the gauge copies and especially all of the Gribov regions in SU(2) within the fluctuation cut-off Λ_A.

Gribov, *Nucl. Phys.* B139 (1978) 1.



- Regions out of the fluctuation cut-off Λ_A might contribute to the path integrals, but cut-off dependence of U(1) and SU(2) results (Lagrangian density, potential, A-distribution) are negligible.

--> Conjecture: gauge-field fluctuation in our approach is within certain "band" [average of A_μ is almost zero in simulations].



--- Fluctuation out of the "band" will be suppressed by the weight exp(-S).