Chiral symmetry restoration with functional renormalization group methods

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Outline

Motivation

- QCD Lagrangian with quarks and gluons:

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu a} + \bar{\psi}_i (i\gamma_\mu D^\mu - m)_{ij} \psi_j$$

- Approximate chiral symmetry:

$$\psi_L \to e^{iT^a \theta^a_L} \psi_L, \qquad \psi_R \to e^{iT^a \theta^a_R} \psi_R$$

$\longrightarrow U_L(N_f) \times U_R(N_f) \sim U_V(N_f) \times U_A(N_f)$
$\longrightarrow$ depending on the energy, $N_f = 2, 3$ have relevance

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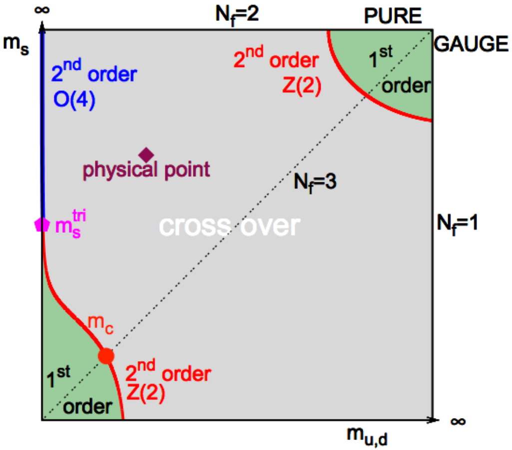
- Chiral symmetry is spontaneuously broken in the ground state:

$$< \bar{\psi}_i \psi_i > \; = \; < \bar{\psi}_{i,R} \psi_{i,L} > + < \bar{\psi}_{i,L} \psi_{i,R} > \; \neq \; 0$$

$\longrightarrow < \bar{\psi}_{i,R} \psi_{j,L} > \sim \delta_{ij} \Rightarrow$ symmetry broken to $U_V(N_f)$

- Chiral symmetry restoration? Critical temperature?
  Quark mass dependence? Axial anomaly?

Motivation



$\longrightarrow$ $N_f = 2$ case: 2nd order nature depends on anomaly strenght

$\longrightarrow$ small anomaly case: subtle, fixed point?

$\longrightarrow$ vanishing quark masses: first order, but no direct evidence

Chiral symmetry in effective meson models

- Lagrangian of the n-flavor low energy strong interaction:

$$\mathcal{L} = \partial_\mu M \partial^\mu M^\dagger - \mu^2 \,\mathrm{Tr}\,(MM^\dagger) - \frac{g_1}{n^2}[\,\mathrm{Tr}\,(MM^\dagger)]^2 - \frac{g_2}{n}\,\mathrm{Tr}\,(MM^\dagger)^2$$

$\longrightarrow$ $M = T^a(s^a + i\pi^a)$ [scalar and pseudoscalar mesons]

$\longrightarrow$ vanishing quark masses

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- Renormalization group analysis: fixed point(s)?

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- Renormalization group analysis: fixed point(s)?
- β functions (ϵ-expansion, 1-loop)

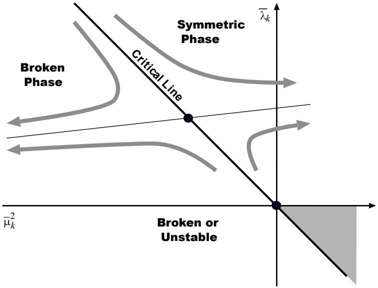
$$\begin{aligned}
\beta_1 &= -\epsilon\bar{g}_1 + \frac{n^2+4}{3}\bar{g}_1^2 + \frac{4n}{3}\bar{g}_1\bar{g}_2 + \bar{g}_2^2 \\
\beta_2 &= -\epsilon\bar{g}_2 + \frac{2n}{3}\bar{g}_2^2 + 2\bar{g}_1\bar{g}_2
\end{aligned}$$

$\longrightarrow$ in 3d, fixed points: $\bar{g}_1 = \frac{3\epsilon}{n^2+4}$, $\bar{g}_2 = 0$ [$O(2n^2)$ W.F.]

$\bar{g}_1 = 0$, $\bar{g}_2 = 0$ [Gaussian]

$\longrightarrow$ no IR stable fixed point exists!

Chiral symmetry in effective meson models



- Inclusion of only the first quartic coupling:
  $\longrightarrow$ $O(2n^2)$ model with second order transition

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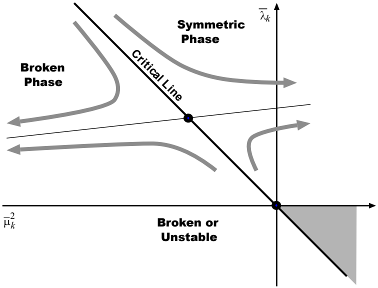

- Inclusion of only the first quartic coupling:
  $\longrightarrow$ $O(2n^2)$ model with second order transition
- Adding the second coupling: RG trajectories diverge from f.p.
  $\longrightarrow$ no second order transition
  $\longrightarrow$ indirect evidence of a first order transition
- Direct evidence?
  $\longrightarrow$ Construction of the finite temperature effective potential

Functional renormalization group flows

- FRG: follows the idea of Wilsonian renormalization group

$$Z_k[J] = \exp(iW_k[J]) = \int \mathcal{D}\phi e^{i\left(S[\phi]+\int J\phi+\int \frac{1}{2}\phi R_k \phi\right)}$$

- R_k: IR regulator function

$\Rightarrow$ requriements: 1., scale separation (suppress modes $q \lesssim k$)

2., $R_k \longrightarrow \infty$ as $k \longrightarrow \infty$

3., $R_k \longrightarrow 0$ as $k \longrightarrow 0$

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  2., $R_k \longrightarrow \infty$ as $k \longrightarrow \infty$
  3., $R_k \longrightarrow 0$ as $k \longrightarrow 0$
- scale-dependent effective action:

$$\Gamma_k[\bar{\phi}] = W_k[J] - \int J\bar{\phi} - \frac{1}{2}\int \bar{\phi} R_k \bar{\phi}$$

$$e^{i\Gamma_k[\bar{\phi}]} = \int \mathcal{D}\phi e^{i\left(S[\phi]+\int \frac{\delta\Gamma_k}{\delta\bar{\phi}}(\phi-\bar{\phi})+\frac{1}{2}\int(\bar{\phi}-\phi)R_k(\bar{\phi}-\phi)\right)}$$

$$\Longrightarrow \Gamma_{k\approx\infty}[\bar{\phi}] = S[\bar{\phi}], \qquad \Gamma_{k=0}[\bar{\phi}] = \Gamma_{1PI}[\bar{\phi}]$$

- scale-dependent effective action interpolates between the classical- and quantum effective action

- The scale-dependent effective action obeys the following flow equation:

$$\partial_k \Gamma_k = \frac{1}{2}\,\mathrm{STr}\left[\frac{1}{\Gamma_k^{(2)}[\bar{\phi}] + R_k}\partial_k R_k\right]$$

$\longrightarrow$ functional integro-differential equation
$\longrightarrow$ not solvable, approximation(s) needed

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- Approximations? $\rightarrow$ derivative expansion!

$$\Gamma_k[\phi] = \int_x \Big(V_k[\phi] + \phi(Z_k\partial_\tau - A_k\nabla^2 - W_k\partial_\tau^2)\phi + ...\Big)$$

$\Rightarrow Z_k, A_k = 1, W_k = 0$ with $V_k \neq 0$ is reliable (LPA)

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- Finite temperature flow equation for the local potential:

$$\partial_k V_k = \frac{k^4}{6\pi^2} T \sum_{\omega_m}\sum_i \frac{1}{\omega_m^2 + k^2 + \mu_i^2(k)}$$

Functional renormalization group flows

- Symmetry breaking pattern (Wafa-Vitten theorem):

  $\Longrightarrow U_L(n) \times U_R(n) \to U_V(n)$

  $\Longrightarrow < M >= v_0 T^0 \sim \hat{\mathbf{1}}$

- V_k is a function of:

  $\Longrightarrow$ chiral invariants:
  $$I_1 = \mathrm{Tr}\,[MM^\dagger]$$
  $$I_2 = \mathrm{Tr}\,[MM^\dagger - \mathrm{Tr}\,(MM^\dagger)/n)]^2$$
  $$I_3 = \mathrm{Tr}\,[MM^\dagger - \mathrm{Tr}\,(MM^\dagger)/n)]^3$$
  $$\ldots$$

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  $$\dots$$
- Chiral expansion around the $< M >$ configuration:

$$V_k(I_1, I_2, \dots I_n) = U_k(I_1) + \sum_{\{\alpha\}} C_k^{(\alpha)}(I_1) \prod_{i=2}^{n} I_i^{\alpha_i}$$

- We derive and solve flow equations for the coefficient functions U_k and $C_k^{(\alpha)}$

  $\longrightarrow$ very efficent numerically

  $\longrightarrow$ 1-dimensional grids (not n-dim.)

Functional renormalization group flows

- Flow equations of the coefficients are similary to the Dyson-Schwinger hierarchy:

$$
\begin{aligned}
U_k(I_1) &\longleftarrow C_k^{(0,1,0,..)} \\
C_k^{(0,1,0...)} &\longleftarrow C_k^{(0,0,1,0...)}, C_k^{(0,2,0,..)} \\
C_k^{(0,0,1,0...)} &\longleftarrow \ldots
\end{aligned}
$$

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$$\begin{array}{rcl} U_k(I_1) & \longleftarrow & C_k^{(0,1,0,..)} \\ C_k^{(0,1,0...)} & \longleftarrow & C_k^{(0,0,1,0...)},\, C_k^{(0,2,0,..)} \\ C_k^{(0,0,1,0...)} & \longleftarrow & ... \end{array}$$

- Truncation is necessary
  $\longrightarrow$ we keep only those coefficients that are already nonzero at classical level

$$V_k \approx U_k(I_1) + C_k^{(0,1,0,0..)}(I_1) \cdot I_2$$

- Evaluation of V_k at $< M >= v_0 T^0$
  $\longrightarrow \qquad I_1|_{v_0} = v_0^2/2, \qquad I_2|_{v_0} = 0$

Functional renormalization group flows

$$\partial_k U_k(I_1) = \frac{k^4 T}{6\pi^2} \sum_{\omega_m} \left(\frac{n^2}{\omega_m^2 + E_\pi^2} + \frac{n^2 - 1}{\omega_m^2 + E_{a_0}^2} + \frac{1}{\omega_m^2 + E_\sigma^2} \right)$$

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$$\begin{aligned}
\partial_k C_k(I_1) = \frac{k^4 T}{6\pi^2} \sum_{\omega_m} \Bigg[& \frac{4(3C_k + 2I_1 C_k')^2/n}{(\omega_m^2 + E_{a_0}^2)^2(\omega_m^2 + E_\sigma^2)} \\
& + \frac{128 C_k^5 I_1^3/n}{(\omega_m^2 + E_\pi^2)^3(\omega_m^2 + E_{a_0}^2)^3} \\
& + \frac{4C_k\left(4C_k(n^2-3) + (1-4n^2)I_1 C_k'\right)/n}{(\omega_m^2 + E_{a_0}^2)^3} \\
& + \frac{4\left(3C_k C_k' I_1 + 4I_1^2 C_k' + C_k(3C_k - 2C_k'' I_1^2)\right)/n}{(\omega_m^2 + E_{a_0}^2)(\omega_m^2 + E_\sigma^2)^2} \\
& + \frac{64 C_k^3 I_1^2 (C_k - I_1 C_k')/n}{(\omega_m^2 + E_\pi^2)^2(\omega_m^2 + E_{a_0}^2)^3} - \frac{48 C_k^2 I_1^2 C_k'}{(\omega_m^2 + E_\pi^2)(\omega_m^2 + E_{a_0}^2)^3} + \ldots
\end{aligned}$$

Functional renormalization group flows

- Assumption of V_k: (form of classical potential)

$$V_k = \mu_k^2 \,\mathrm{Tr}\,(MM^\dagger) + \frac{g_{1,k}}{n^2}[\,\mathrm{Tr}\,(MM^\dagger)]^2 + \frac{g_{2,k}}{n}\,\mathrm{Tr}\,(MM^\dagger)^2$$

- We recover the one-loop β-functions:

$$\begin{aligned}\beta_1 &= -\epsilon\bar{g}_{1,k} + \frac{n^2+4}{3}\bar{g}_{1,k}^2 + \frac{4n}{3}\bar{g}_{1,k}\bar{g}_{2,k} + \bar{g}_{2,k}^2 \\ \beta_2 &= -\epsilon\bar{g}_{2,k} + \frac{2n}{3}\bar{g}_{2,k}^2 + 2\bar{g}_{1,k}\bar{g}_{2,k}\end{aligned}$$

- Flow of the mass parameter:

$$\partial_k \mu_k^2 = -k^4 \frac{(n^2+1)g_{1,k} + 2n g_{2,k}}{6(k^2+\mu_k^2)^2}$$

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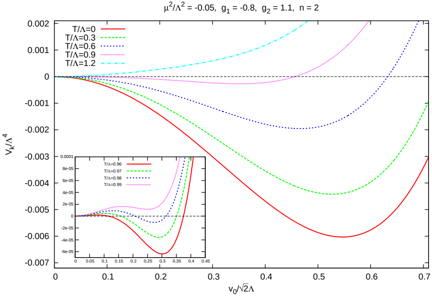
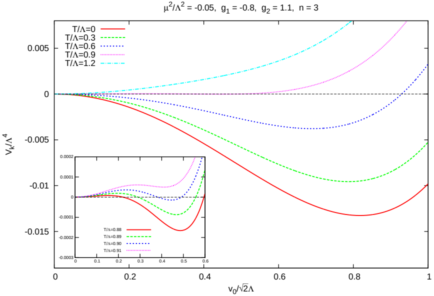
- The functional flow equations contain much more:
  $\longrightarrow$ infinite resummation of n-point couplings

Numerical results (effective potential)





- $k = 0$ is very demanding to reach numerically
  $\longrightarrow$ the flow was stopped at $k/\Lambda = 0.2$
  $\longrightarrow$ at finite k, the potential is not convex
- First order transition is observed
  $\longrightarrow$ in the whole range of the parameter space
  $\longrightarrow$ irrespectively of the flavor number n

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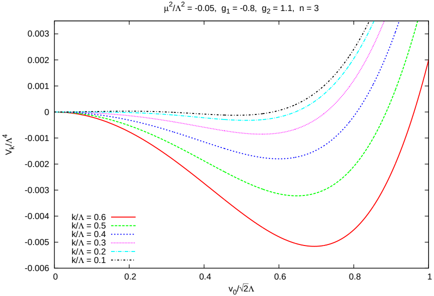
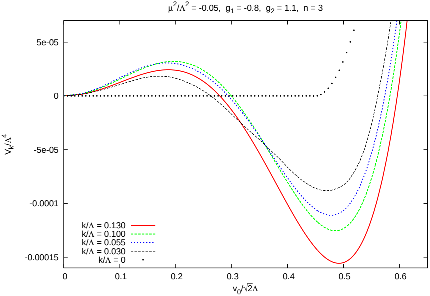


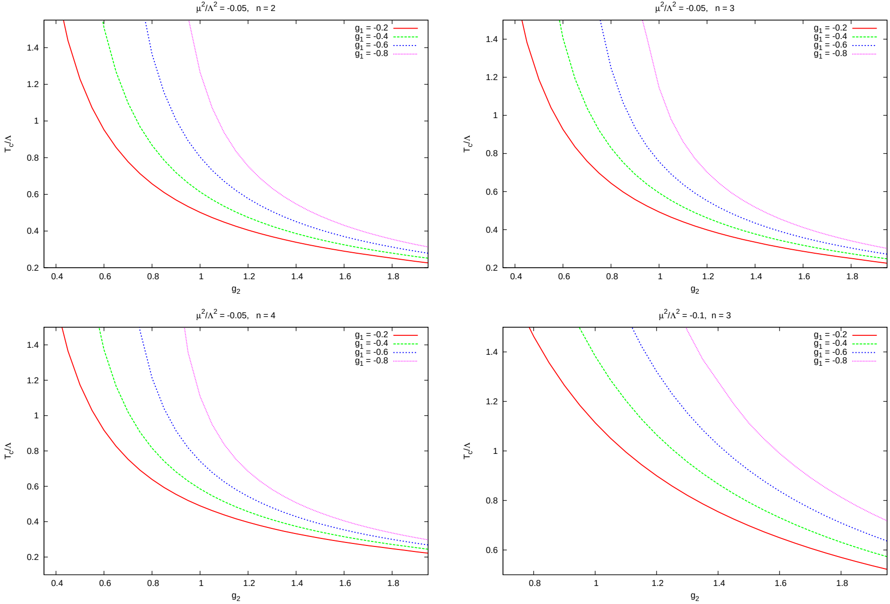


- When $k \to 0$, the potential gradually becomes convex
- The effective potential is not a convenient quantity for identifying 1st order transitions
  $\longrightarrow$ crit. temp. and discontinuation are defined as limits:

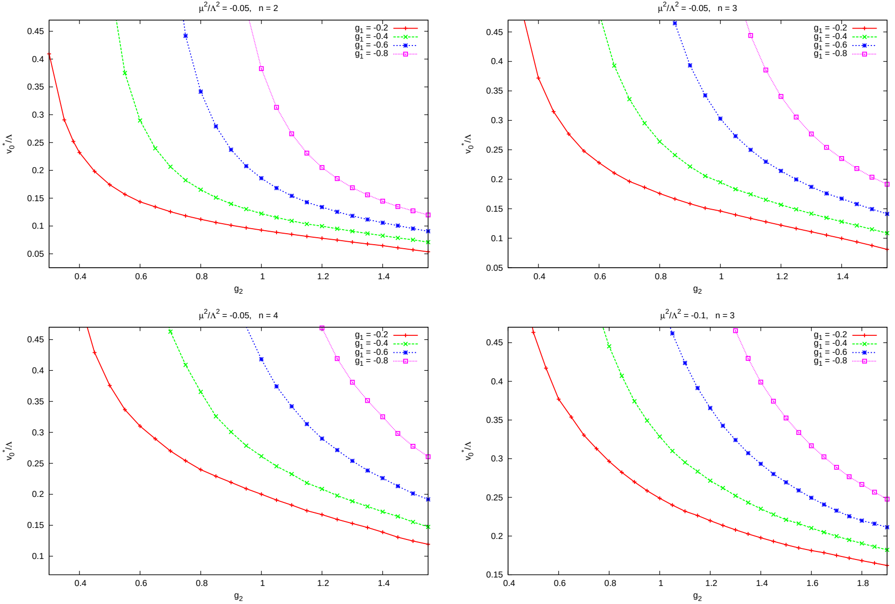
$$T_C = lim_{k\to 0} T_C(k), \qquad \Delta v_0 = lim_{k\to 0}\Delta v_0(k)$$

$\longrightarrow$ numerics: they can be obtained via extrapolation

Numerical results (T_C)



Numerical results (discontinuity)

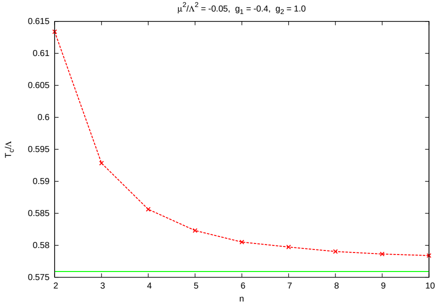


Numerical results (large-n)

- Large-n scalings of functions:

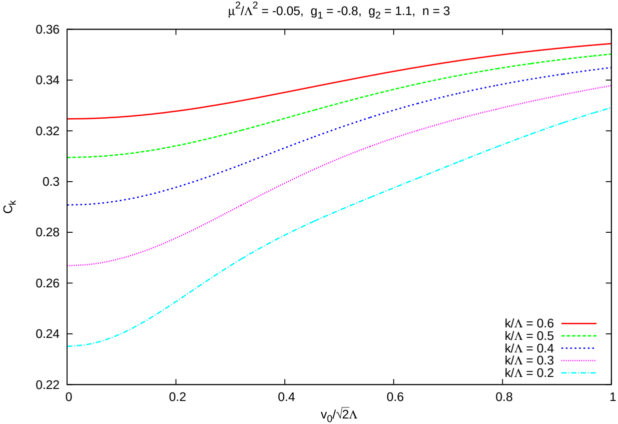
$$U_k(I_1) = n^2 u_k(i_1), \qquad C_k(I_1) = c_k(i_1)/n, \qquad I_1 = n^2 i_1$$

- n-dependence of T_C:



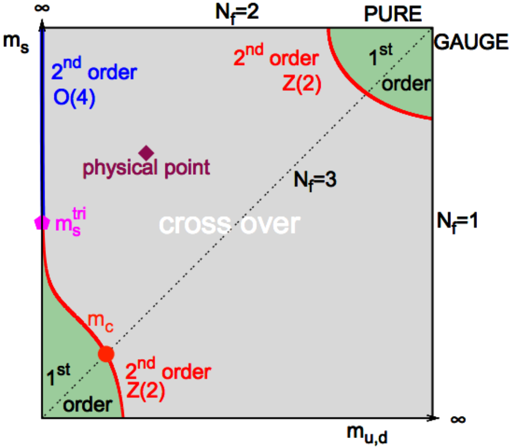
- Less than 3% difference between $n = 3$ and $n = \infty$
$\longrightarrow$ large-n expansion is quite robust

Numerical results



- Approximating $C_k(I_1) \approx const.$ is crude $\longrightarrow$ the function develops a structure as $k \to 0$

Numerical results



$\longrightarrow$ our method: no anomaly included

$\longrightarrow$ for $N_f = 2, 3$ we obtain first order transitions

$\longrightarrow$ if anomaly disappears at $T_C \Rightarrow$ Columbia plot has to change

Anomaly

- Anomaly term in the Lagrangian:

$$\mathcal{L}_{U_A(1)} = c(\det M + \det M^\dagger)$$

$\longrightarrow$ changes the masses and spoils chiral symmetry

$$\partial_k V_k = \frac{k^4}{6\pi^2} T \sum_{\omega_m} \sum_i \frac{1}{\omega_m^2 + k^2 + \mu_i^2(k)}$$

$\longrightarrow V_k \neq V_k(I_1, I_2, \ldots)$

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$\longrightarrow$ $V_k \neq V_k(I_1, I_2, ...)$

- Way out: expand the r.h.s. in terms of the anomaly coefficient
- This procedure is compatible with the Ansatz

$$V_k = U_k(I_1) + C_k(I_1) \cdot I_2 + c_k(I_1)(\det M + \det M^\dagger)$$

$\longrightarrow$ obtain the flow and T-dependence of $c_k(I_1)$ coeff.

Finite quark masses

- Finite quark masses are realized as explicit symmetry breaking terms:

$$\mathcal{L}_h = \mathrm{Tr}\left[H(M + M^\dagger)\right] \equiv h_0 s^0 + h_8 s^8$$

$\longrightarrow$ these couplings do not change the flow equations at all
$\longrightarrow$ they do not have an RG-flow
$\longrightarrow$ only effect: shift the value of the effective potential

- Implementation is easy and straightforward

Work is under progress...

Conclusions

- Analysis of the $U(n) \times U(n)$ meson model
  $\longrightarrow$ no anomaly, zero quark masses
  $\longrightarrow$ top left and bottom left regions of the Columbia plot
- Functional renormalization group method
  $\longrightarrow$ local potential approximation
  $\longrightarrow$ chiral invariant expansion
- Calculation of the effective potential
  $\longrightarrow$ convexity
  $\longrightarrow$ $T_C = \lim_{k\to 0} T_C(k)$, $\Delta v_0 = \lim_{k\to 0} \Delta v_0(k)$
- Only first order transitions have been observed, irrespectively of n
  $\longrightarrow$ if the anomaly is recovered around T_C,
  no second order transition appears!
  $\longrightarrow$ Columbia plot changes