Wigner and Husimi distributions in QCD

Yoshitaka Hatta

(Yukawa institute, Kyoto U.)

Nucl.Phys. A940 (2015) 158; with Yoshikazu Hagiwara (萩原慶一）

Outline

- Introduction
- Wigner distribution in Quantum Mechanics and QCD
- Husimi distribution in QM and QCD
- Discussions

3D tomography of the nucleon

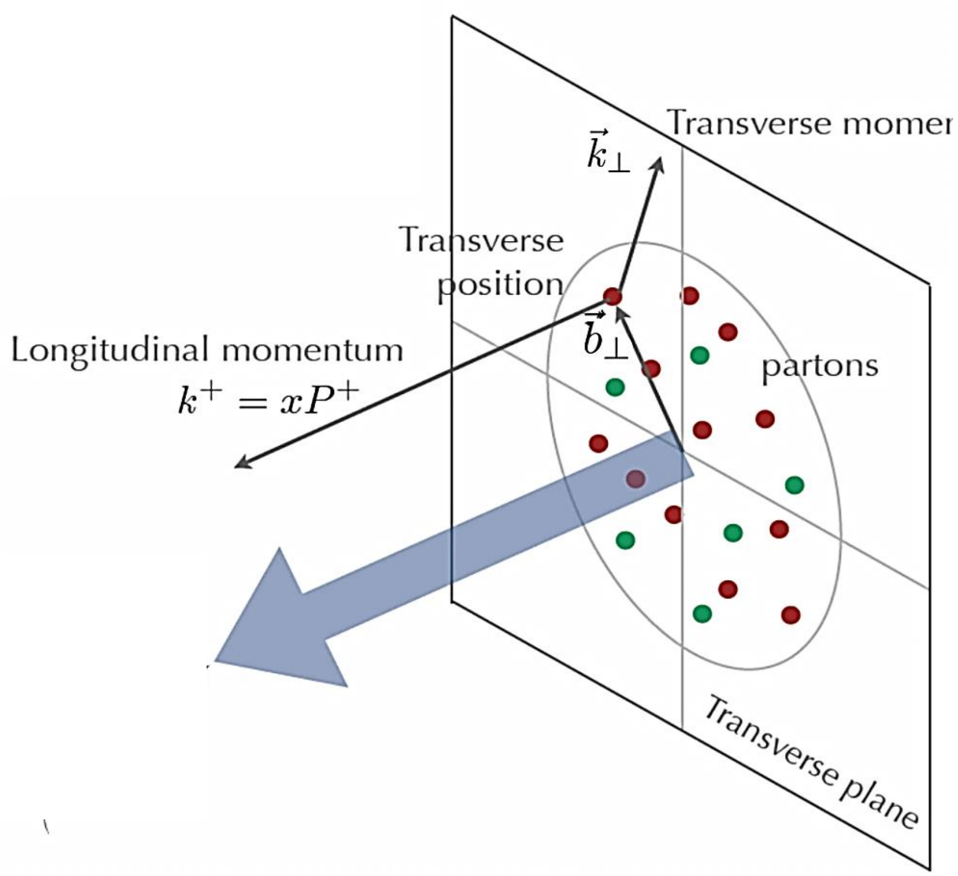
Partons inside the nucleon are characterized not only by the longitudinal momentum fraction x

TMD PDF $f(x, \vec{k}_\perp)$

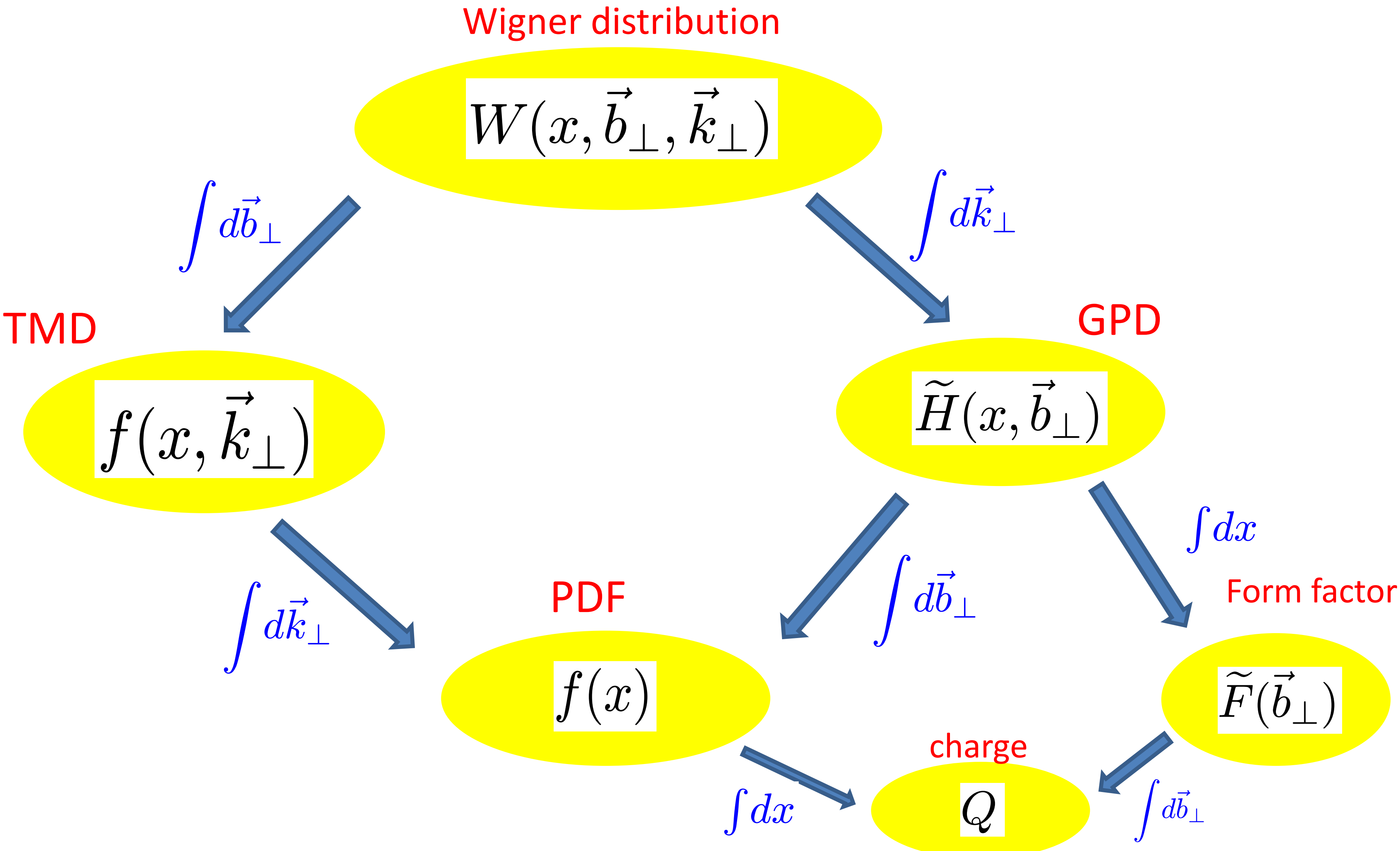
GPD $H(x, \vec{\Delta}_\perp)$

F.T. $\updownarrow$

$\widetilde{H}(x, \vec{b}_\perp)$



5D tomography:
Wigner distribution— the “mother distribution”



Wigner distribution in QM (1932)

Phase space distribution in quantum mechanics

$$f_W(q,p) = \int_{-\infty}^{\infty} dx e^{-ipx/\hbar} \langle\psi|q - x/2\rangle\langle q + x/2|\psi\rangle$$

$$= \int_{-\infty}^{\infty} dx e^{-ipx/\hbar} \langle q + x/2|\hat{\rho}|q - x/2\rangle$$

density matrix $\hat{\rho} = |\psi\rangle\langle\psi|$

Eugene Wigner (1902-1995)

Moments of the Wigner distribtuion

$$\int \frac{dq}{2\pi\hbar} f_W(q,p) = |\langle\psi|p\rangle|^2 \,, \quad \int \frac{dp}{2\pi\hbar} f_W(q,p) = |\langle\psi|q\rangle|^2$$

Expectation value of an operator $\langle\hat{\mathcal{O}}\rangle = \int dp dq f_W(q,p) \mathcal{O}(p,q)$

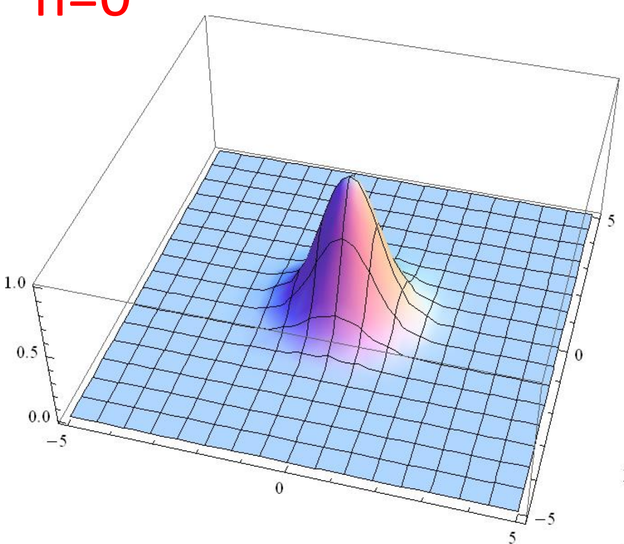
Wigner distribution for the harmonic oscillator

Laguerre polynomial

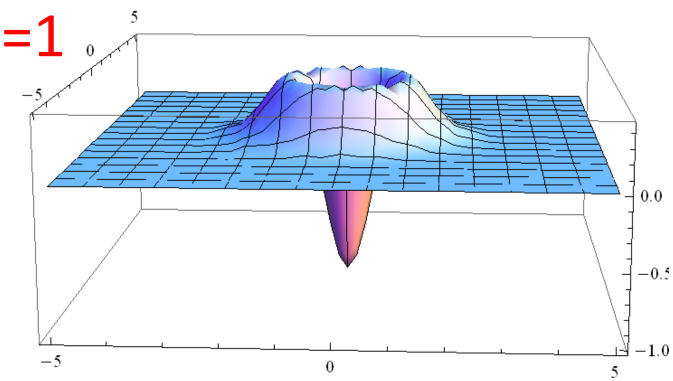
$$H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}$$

$$f_W(q,p) = 2(-1)^n e^{-2H/\hbar\omega} L_n\left(\frac{4H}{\hbar\omega}\right)$$

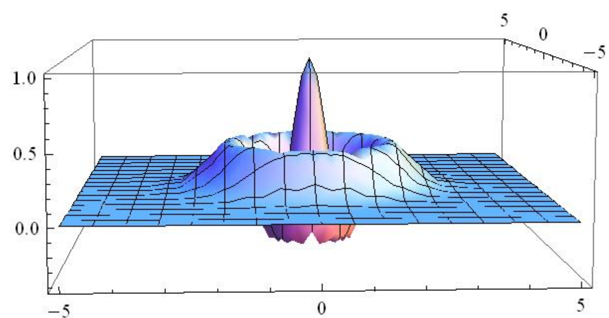
n=0



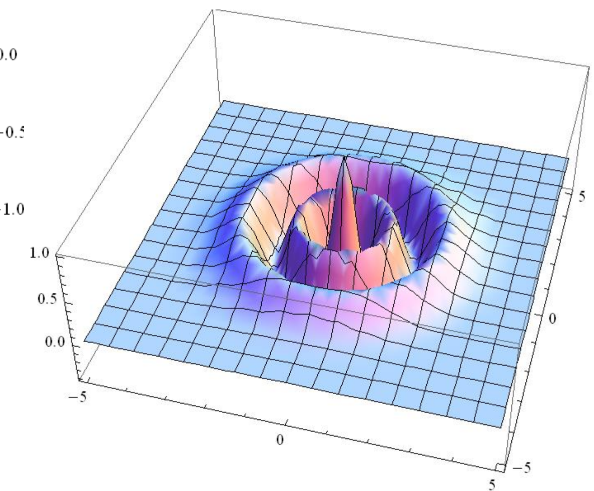
n=1



n=2



n=4



Probability distribution?
No way!

The uncertainty principle

$$\Delta q \Delta p \geq \frac{\hbar}{2}$$

The very notion of “phase space distribution” in quantum physics contradicts the uncertainty principle.

→ Wigner distribution wildly oscillates and becomes negative.
Incorporates (interesting) quantum interference effect,
but no probabilistic interpretation

Wigner distribution in QCD

Ji (2003)
Belitsky, Ji, Yuan (2003)

Wigner distribution of quarks in the nucleon

$$W^{\Gamma}(x,\vec{b}_\perp,\vec{k}_\perp)$$
$$=\int \frac{dz^- d^2 z_\perp}{16\pi^3}\frac{d^2\Delta_\perp}{(2\pi)^2}e^{i(xP^+z^- - \vec{k}_\perp\cdot\vec{z}_\perp)}\langle P+\frac{\Delta}{2}|\bar{q}(b-\frac{z}{2})\Gamma\mathcal{L}\,q(b+\frac{z}{2})|P-\frac{\Delta}{2}\rangle$$

$\vec{b}_\perp$ integral → TMD

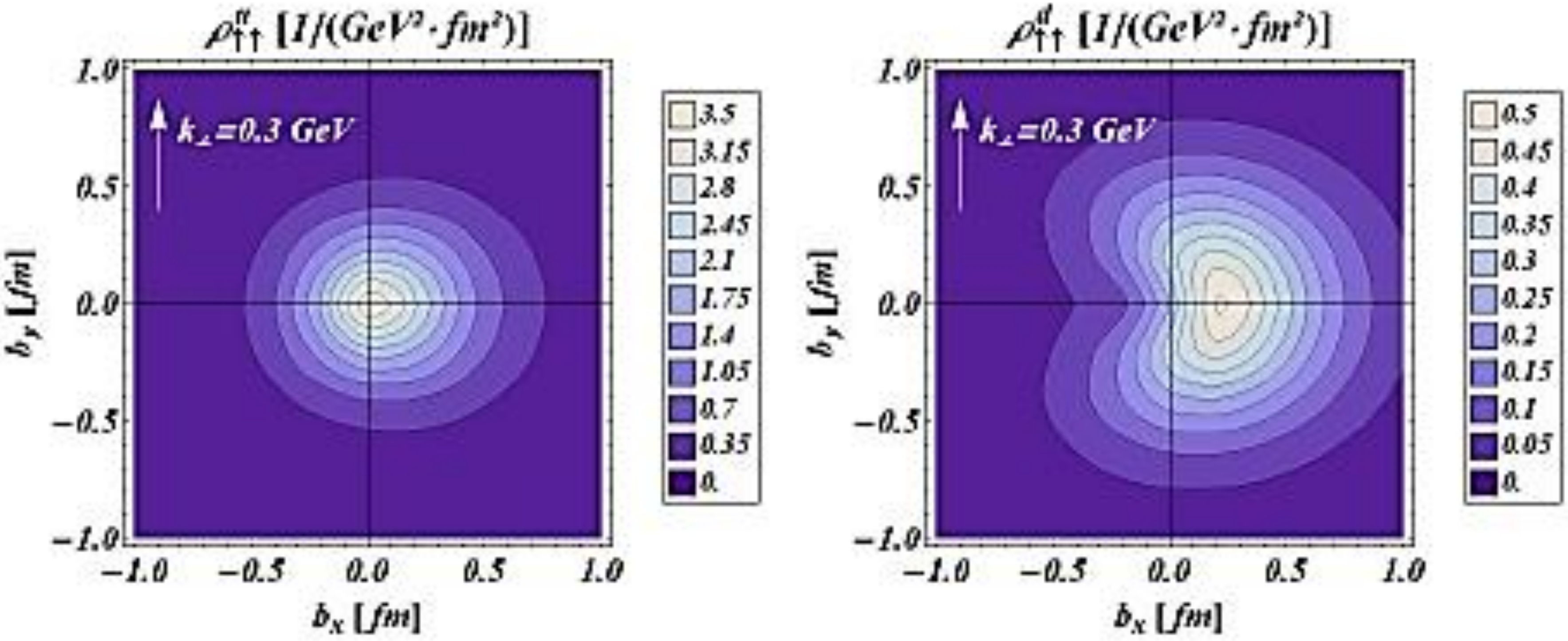
$\vec{k}_\perp$ integral → GPD

$$\Delta^\mu=(0,0,\vec{\Delta}_\perp)$$

momentum recoil
(relativistic effect)

Model calculation

Lorce, Pasquini, (2011)



light-cone quark models
(no gluons included)

Relation to Orbital Angular Momentum (OAM)

Lorce, Pasquini, (2011)
YH (2011)

Jaffe-Manohar decomposition of the nucleon spin

$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma + \Delta G + L^q_{can} + L^g_{can}$$

$$L^q_{can} \sim \bar{\psi}\vec{b} \times i\vec{\partial}\psi$$ → Can be made gauge invariant

Canonical OAM from the Wigner distribution

U-shaped Wilson line

$$L_{can} = \int dx \int d^2 b_\perp d^2 k_\perp (\vec{b}_\perp \times \vec{k}_\perp) W^{\gamma^+}(x, \vec{b}_\perp, \vec{k}_\perp)$$

$$\sim \langle \bar{\psi}\vec{b} \times i\vec{D}_{pure}\psi \rangle$$

$$D^\mu_{pure} = D^\mu - \frac{i}{D^+}F^{+\mu}$$

Husimi distribution (1940)

$$f_H(q,p) = \frac{1}{\pi\hbar}\int dq' dp' e^{-m\omega(q'-q)^2/\hbar-(p'-p)^2/m\omega\hbar} f_W(q',p')$$

Gaussian smearing of the Wigner distribution within the region of minimum uncertainty

$$\Delta q = \sqrt{\hbar/2m\omega} \qquad \Delta p = \sqrt{\hbar m\omega/2}$$

$$\Delta q \Delta p = \hbar/2$$

Kodi Husimi (1909ー2008)

伏見康治

Husimi distribution is positive!

$$f_H(q,p) = \langle\lambda|\hat{\rho}|\lambda\rangle = |\langle\psi|\lambda\rangle|^2 \geq 0$$

Positive semi-definite!

Coherent state

$$a|\lambda\rangle = \lambda|\lambda\rangle$$

$$|\lambda\rangle = e^{-|\lambda|^2/2} e^{\lambda a^\dagger}|0\rangle$$

$$\lambda = \frac{m\omega q + ip}{\sqrt{2\hbar m\omega}}$$

Coherent state satisfies the minimum uncertainty relation

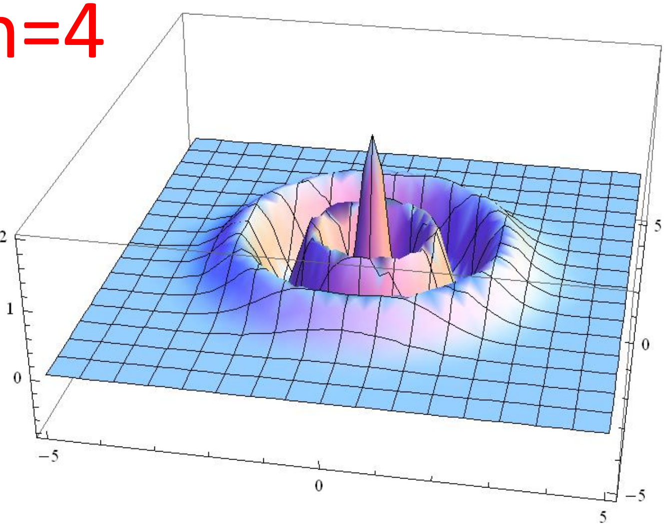
$$\Delta q \Delta p = \hbar/2$$

“Most classical” quantum state

Many applications in statistical physics, laser, chaos, etc.

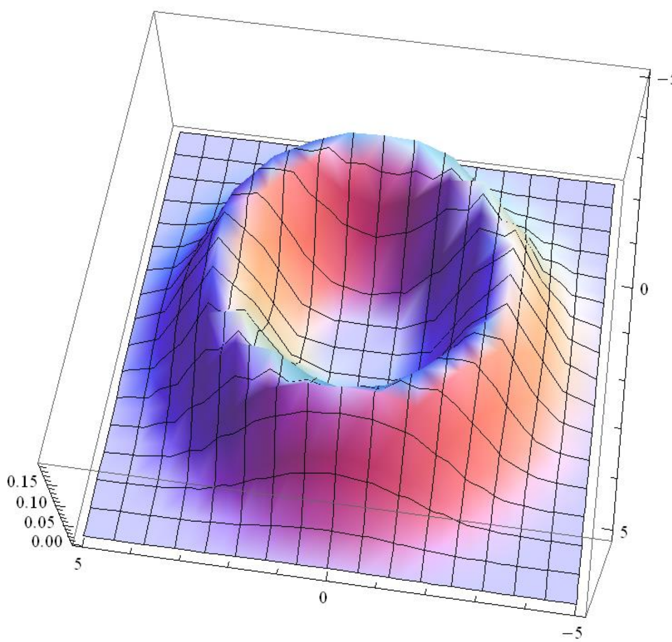
Husimi distribution for the harmonic oscillator

n=4



$$f_W(q,p) = 2(-1)^n e^{-2H/\hbar\omega} L_n\left(\frac{4H}{\hbar\omega}\right)$$

$$f_H(q,p) = \frac{1}{n!} e^{-\frac{H}{\hbar\omega}} \left(\frac{H}{\hbar\omega}\right)^n$$



Localized around the classical orbit

$$H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2} \approx \hbar\omega(n + \frac{1}{2})$$

Husimi distribution in QCD

Hagiwara and YH (2015)

Define

$$H^{\Gamma}(x,\vec{b}_\perp,\vec{k}_\perp)$$
$$\equiv \frac{1}{\pi^2}\int d^2b'_\perp d^2k'_\perp e^{-\frac{1}{\ell^2}(\vec{b}_\perp-\vec{b}'_\perp)^2-\ell^2(\vec{k}_\perp-\vec{k}'_\perp)^2}W^{\Gamma}(x,\vec{b}'_\perp,\vec{k}'_\perp)$$

The parameter ℓ is arbitrary, but it is natural to take $\ell \lesssim R_{hadron}$

Positivity?

In the $A^+ = 0$ gauge

$$H \sim \int d^2\Delta_\perp \, e^{-i\vec{\Delta}_\perp \cdot \vec{b}_\perp - \frac{\ell^2 \Delta_\perp^2}{4}}$$

$$\times \langle P+\Delta/2 | q_+^\dagger \delta(K^+ - (1-x)p^+) e^{-\ell^2 (\vec{K}_\perp + \vec{k}_\perp)^2} \, q_+ | P-\Delta/2 \rangle$$

"good component"

Positive definite if it were not for the momentum recoil Δ (relativistic effect)

However, the Gaussian factor suppresses Δ

Moments of the Husimi distribution

The b-moment of the QCD Husimi distribution does not reduce to the TMD.

$$\int d^2 b_\perp H^\Gamma(x, \vec{b}_\perp, \vec{k}_\perp)$$

$$= \int \frac{dz^- d^2 z_\perp}{16\pi^3} e^{i(xp^+ z^- - \vec{k}_\perp \cdot \vec{z}_\perp)} \underline{e^{-\frac{z_\perp^2}{4\ell^2}}} \langle P|\bar{q}(-z/2)\Gamma\mathcal{L}\, q(z/2)|P\rangle$$

??

Double moments are the same as in the Wigner case

$$\int d^2 b_\perp d^2 k_\perp H^{\gamma^+}(x, \vec{b}_\perp, \vec{k}_\perp) = f(x)$$ (PDF)

$$\int d^2 b_\perp d^2 k_\perp (\vec{b}_\perp \times \vec{k}_\perp) H^{\gamma^+}(x, \vec{b}_\perp, \vec{k}_\perp) = L_{can}$$ (canonical OAM)

Perturbative calculation

Wigner distribution for an on-shell quark

$$W(x,\vec{b}_\perp,\vec{k}_\perp)$$
$$=\int \frac{dz^- d^2 z_\perp}{16\pi^3}\frac{d^2\Delta_\perp}{(2\pi)^2} e^{i(xP^+ z^- - \vec{k}_\perp\cdot\vec{z}_\perp)}\langle P+\frac{\Delta}{2}|\bar{q}(b-\frac{z}{2})\gamma^+\mathcal{L}\, q(b+\frac{z}{2})|P-\frac{\Delta}{2}\rangle$$

Zeroth order

$$W[x,\vec{b}_\perp,\vec{k}_\perp]=\delta(x-1)\delta^{(2)}(\vec{b}_\perp)\delta^{(2)}(\vec{k}_\perp)$$

$\vec{b}_\perp=\vec{k}_\perp=0$!?

Violates the uncertainty principle

$$\Longrightarrow H(x,\vec{b}_\perp,\vec{k}_\perp)=\delta(1-x)\frac{e^{-b_\perp^2/\ell^2-\ell^2 k_\perp^2}}{\pi^2}$$

First order in α_s

Mukherjee, Nair, Ojha, 1403.6233

Bad convergence,
sensitive to $\Delta_\perp^{max}$
divergent when $\vec{b}_\perp = 0$
oscillates in $b_\perp$

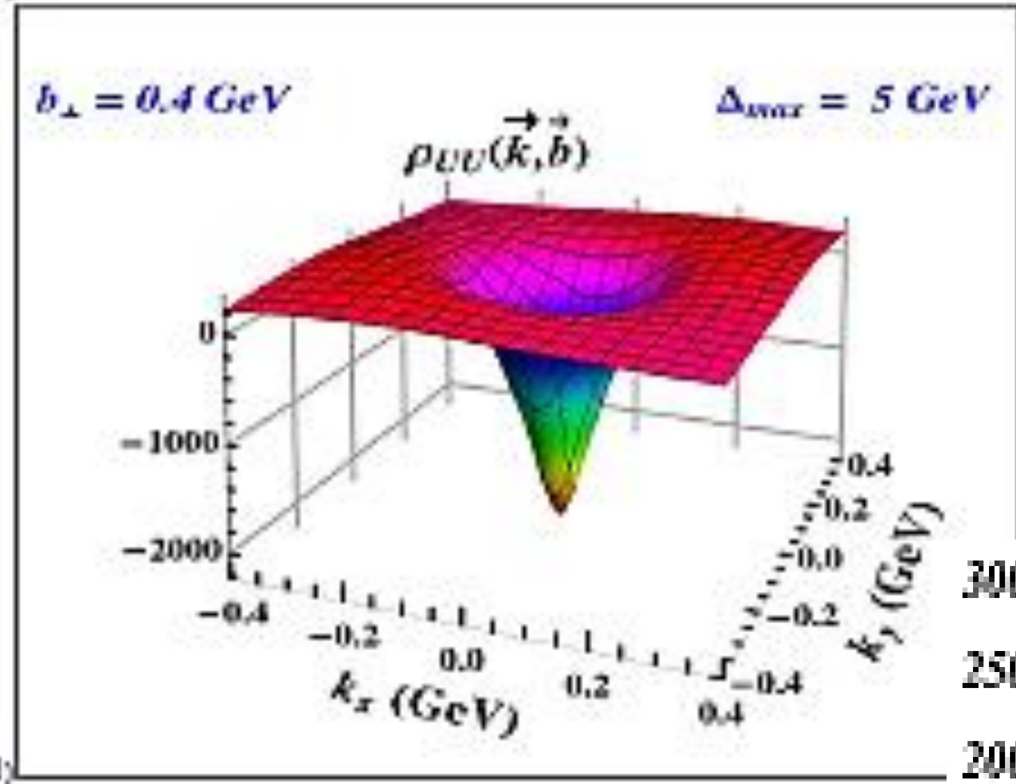
$$W^{\gamma^+}[x, \vec{b}_\perp, \vec{k}_\perp] = \frac{\alpha_s C_F}{2\pi^2} \int \frac{d^2\Delta_\perp}{(2\pi)^2} e^{-i\vec{\Delta}_\perp \cdot \vec{b}_\perp}$$

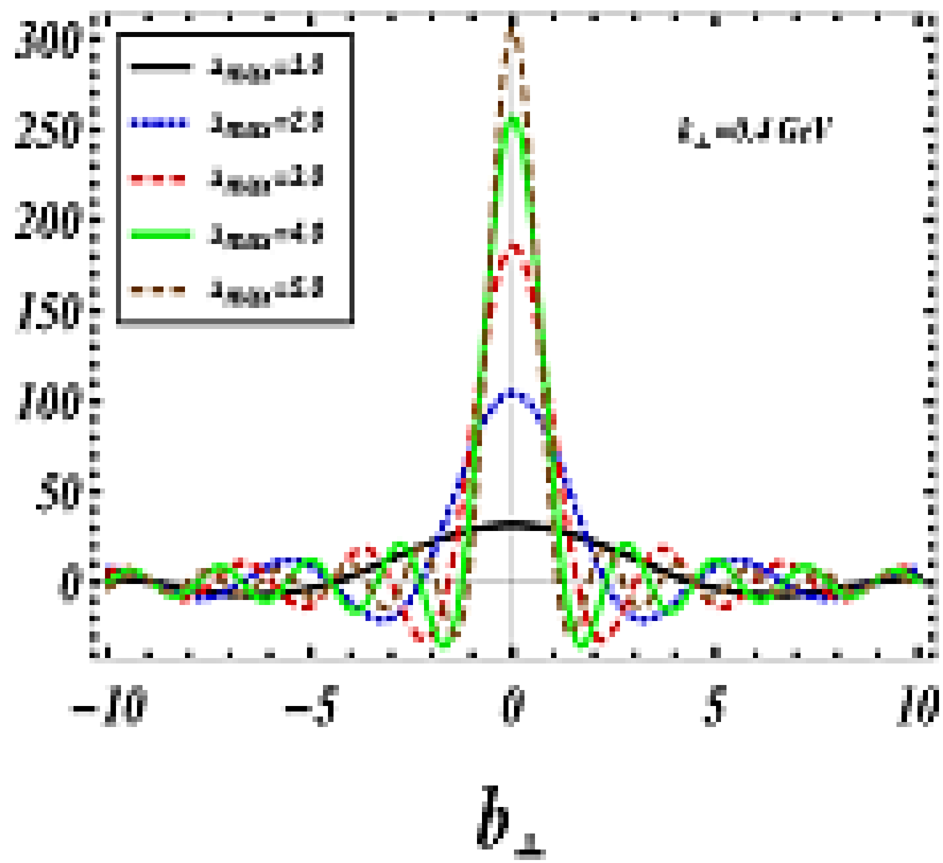
$$\times \frac{\left(k_\perp^2 - \frac{\Delta_\perp^2}{4}(1-x)^2\right) P_{qq}(x) + m^2(1-x)^3}{(q_+^2 + m^2(1-x)^2)(q_-^2 + m^2(1-x)^2)}$$

Negative when
$|\vec{k}_\perp| < (1-x)\frac{|\vec{\Delta}_\perp|}{2}$

splitting function

$$P_{qq}(x) = \frac{1+x^2}{1-x} \qquad \vec{q}_\pm = \vec{k}_\perp \pm \frac{\vec{\Delta}_\perp}{2}(1-x)$$





Mukherjee, Nair, Ojha, 1403.6233

One-loop Husimi distribution

Integration region limited to $|\vec{\Delta}_\perp| < 2/\ell$

Smearing in $|\vec{k}_\perp - \vec{k}'_\perp| \sim 1/\ell$

$$H^{\gamma^+}[x, \vec{b}_\perp, \vec{k}_\perp] = \ell^2 \frac{\alpha_s C_F}{2\pi^3} \int d^2k'_\perp e^{-\ell^2(\vec{k}_\perp - \vec{k}'_\perp)^2} \int \frac{d^2\Delta_\perp}{(2\pi)^2} \cos(\vec{\Delta}_\perp \cdot \vec{b}_\perp) e^{-\frac{\ell^2}{4}\Delta_\perp^2}$$

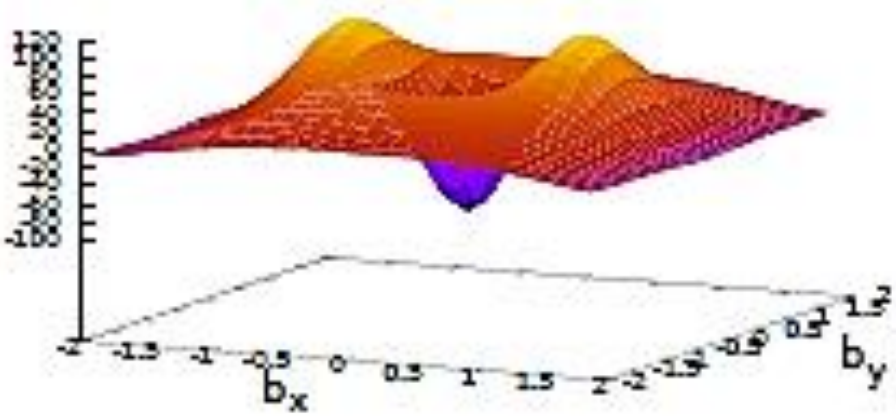
$$\times \frac{\left((k'_\perp)^2 - \frac{\Delta_\perp^2}{4}(1-x)^2\right) P_{qq}(x) + m^2(1-x)^3}{((q'_+)^2 + m^2(1-x)^2)((q'_-)^2 + m^2(1-x)^2)}$$

$$k'_\perp \sim (1-x)\frac{|\vec{\Delta}_\perp|}{2} \cdot \quad \frac{|\vec{\Delta}_\perp|}{2} \cdot \quad \frac{1}{\ell}$$

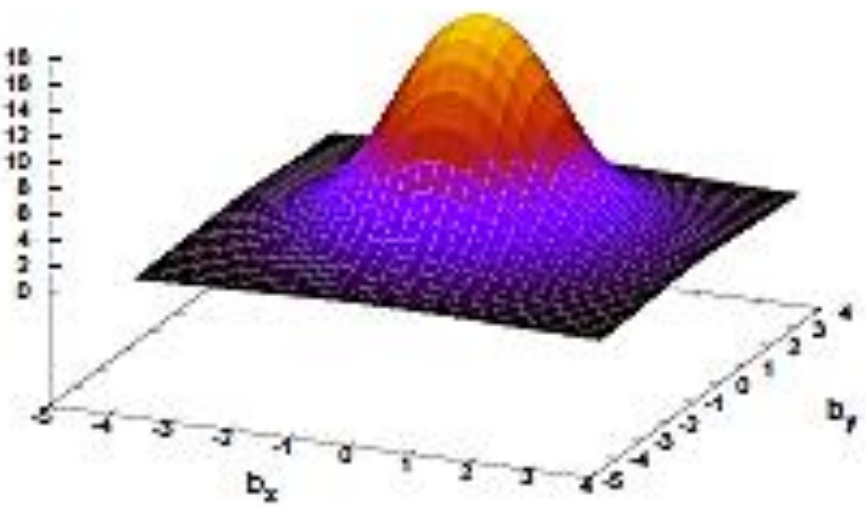
Smearing region larger than the negative region

Numerical result

Before (Wigner)



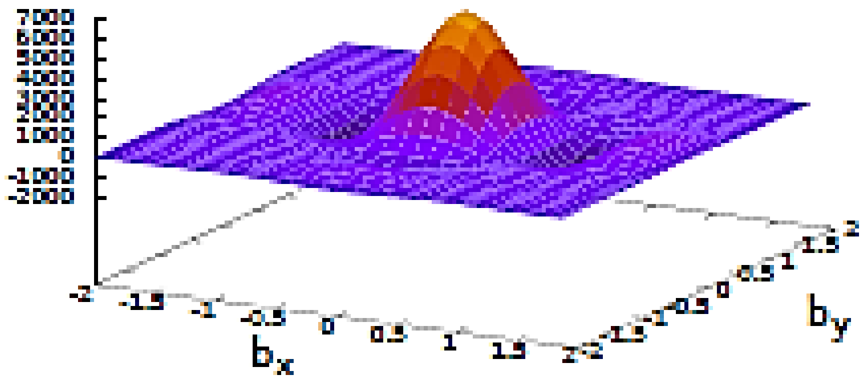
After (Husimi)

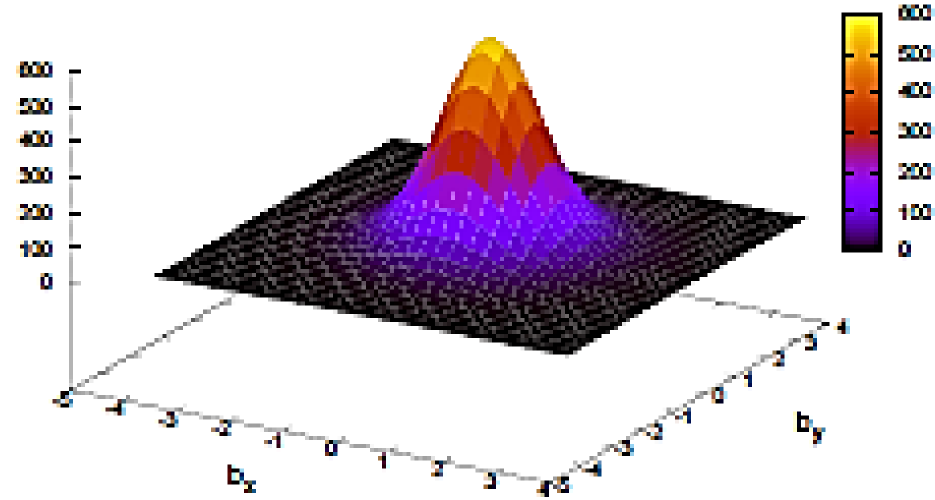


$$x = 0.5,\ m^2 = 0.1\,\mathrm{GeV}^2,\ \ell = 1\,\mathrm{GeV}^{-1}$$

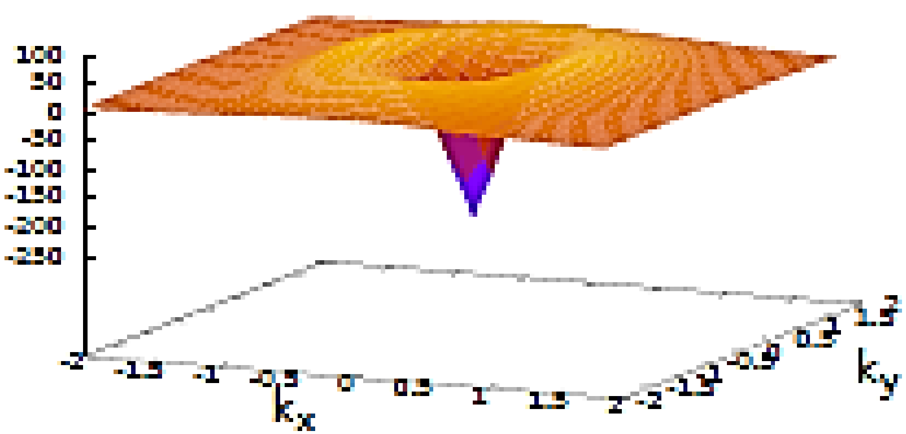
Before (Wigner) After (Husimi)

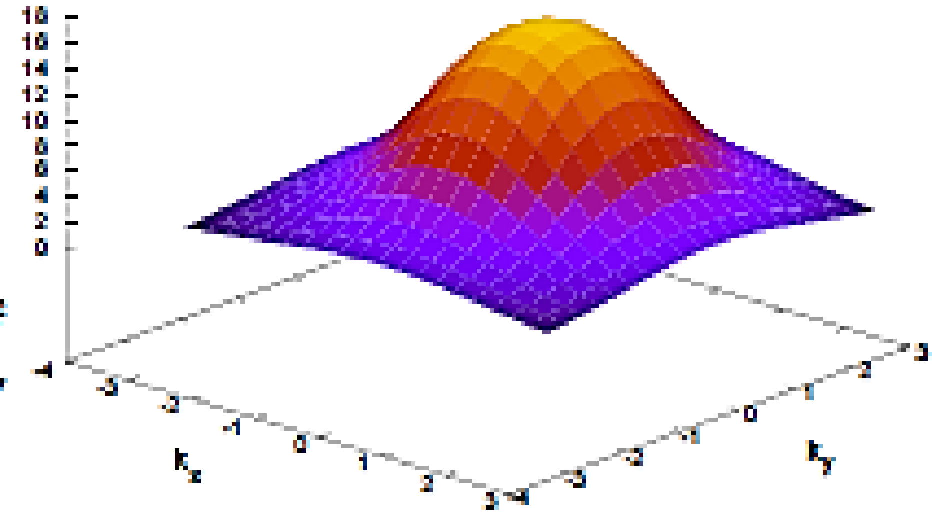
$x = 0.9$





$x = 0.5$





Entropy?

Since the Husimi distribution is positive, one can define entropy (Husimi-Wehrl entropy)

$$S \equiv -\int \frac{dqdp}{2\pi\hbar} f_H \ln f_H$$

cf. von Neumann entropy $S = -tr\hat{\rho}\ln\hat{\rho}$

Nonvanishing even for a pure state.

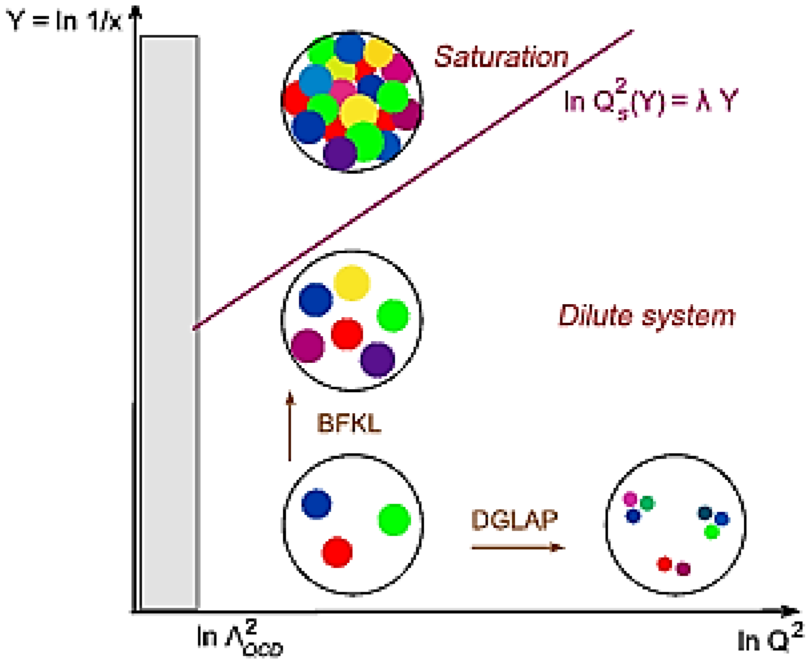
→ A measure of complexity (chaoticity) of the nucleon wavefunction.

Relation to Color Glass Condensate?

- At small-x, the gluons can be treated as a classical coherent state

McLerran, Venugopalan (1993)

- Husimi distribution is the coherent state expectation value.



Any relation between the two?

A tantalizing hint

Recall that the b-moment of the Husimi distribution is not exactly a TMD PDF.

$$\int d^2b_\perp H^\Gamma(x, \vec{b}_\perp, \vec{k}_\perp)$$

$$= \int \frac{dz^- d^2z_\perp}{16\pi^3} e^{i(xp^+z^- - \vec{k}_\perp \cdot \vec{z}_\perp)} \underline{e^{-\frac{z_\perp^2}{4\ell^2}}} \langle P|\bar{q}(-z/2)\Gamma\mathcal{L}\, q(z/2)|P\rangle$$

At low-x, identify $\ell \leftrightarrow \frac{1}{Q_s(x)}$ saturation scale

$$e^{-z_\perp^2/4\ell^2} \to e^{-Q_s^2 z_\perp^2/4}$$ "dipole S-matrix"

What is computed within CGC/quasi-classical approximation could be interpreted as the Husimi distribution.

Summary

- Wigner distribution is often badly-behaved. No probabilistic interpretation.
- Husimi distribution much better behaved, can be interpreted as a probability distribution.
- Proof of positivity hindered by the relativistic kinematical effect, yet a model calculation shows no sign of negative regions.