QCD Evolution of Transverse Momentum Dependent Parton Distributions

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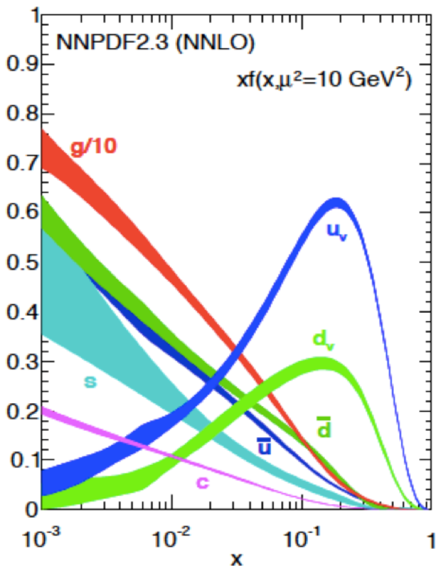
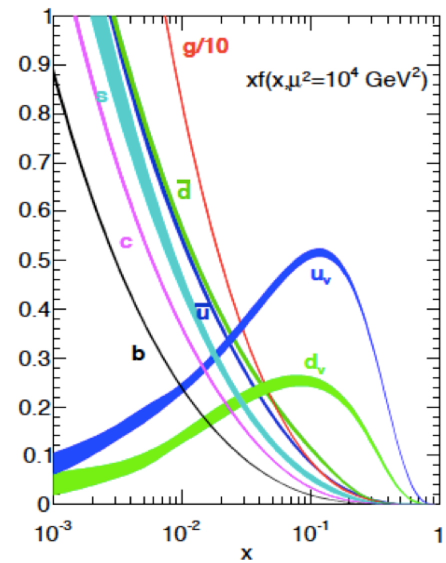
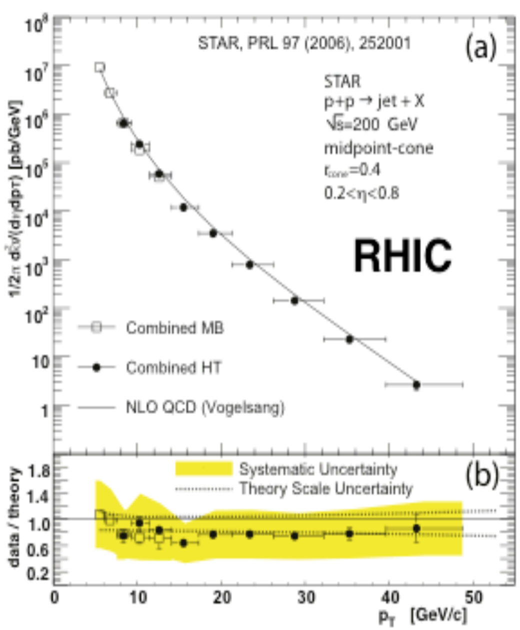
Successes of QCD factorization

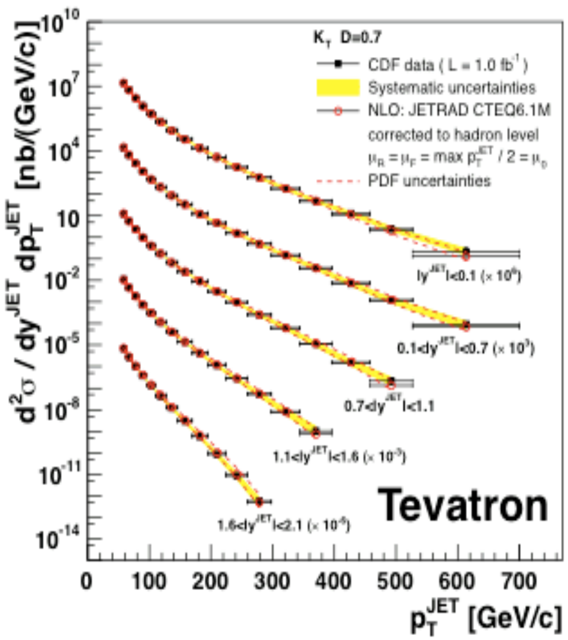


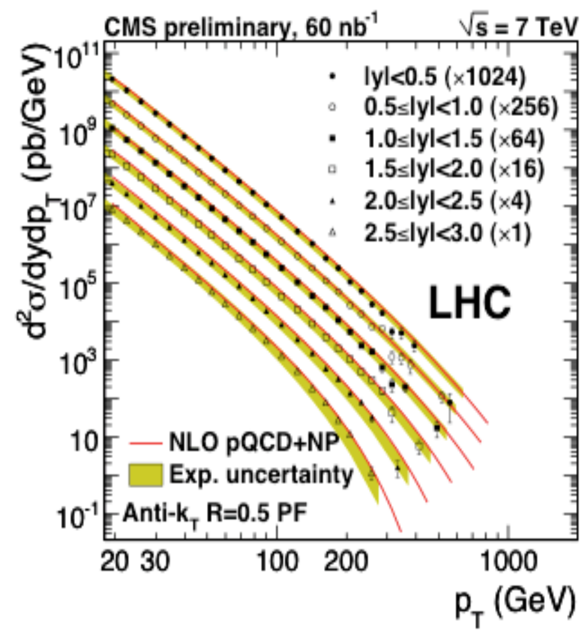
Given PDFs @ μ^2

DGLAP gives PDFs
@
any other scale



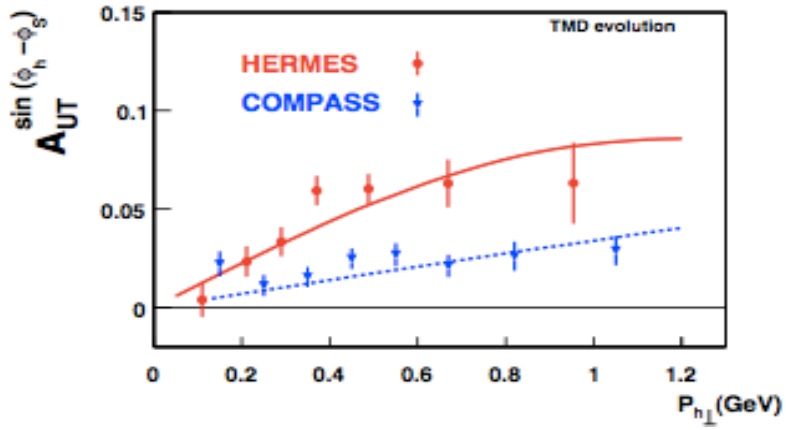


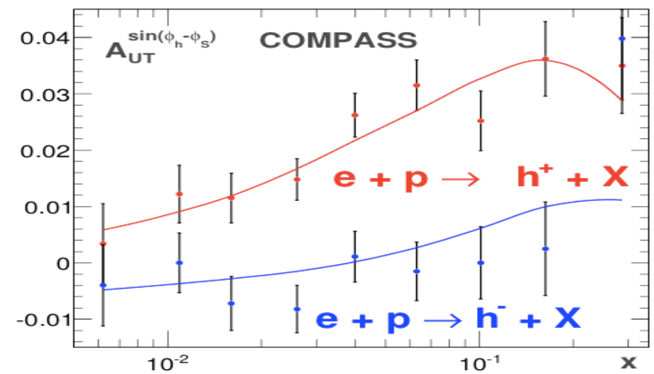




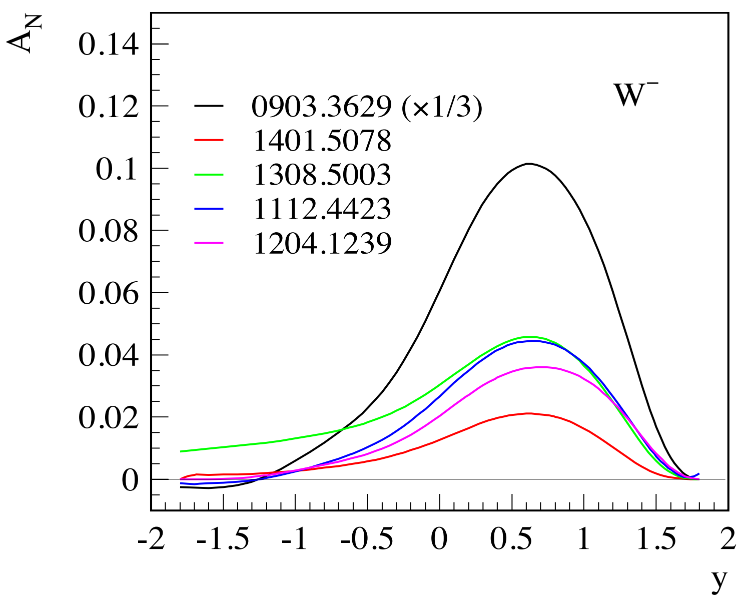
A different story for TMDs

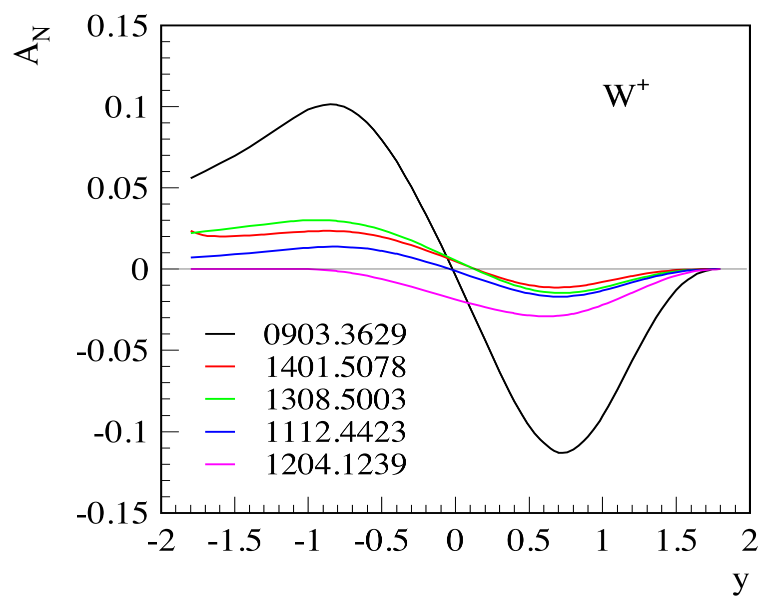
- Fit the same low energy data – Sivers function:





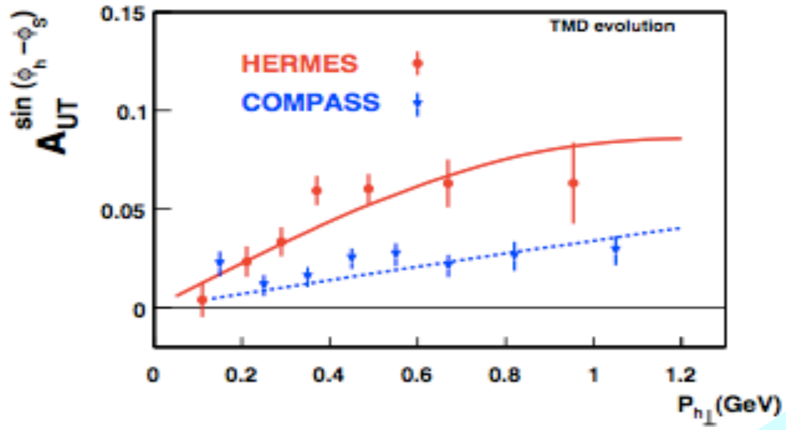
- Very different "predictions" for A_N at a higher energy:

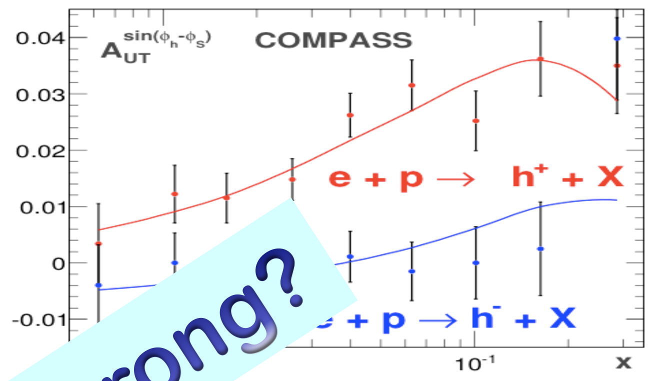




A different story for TMDs

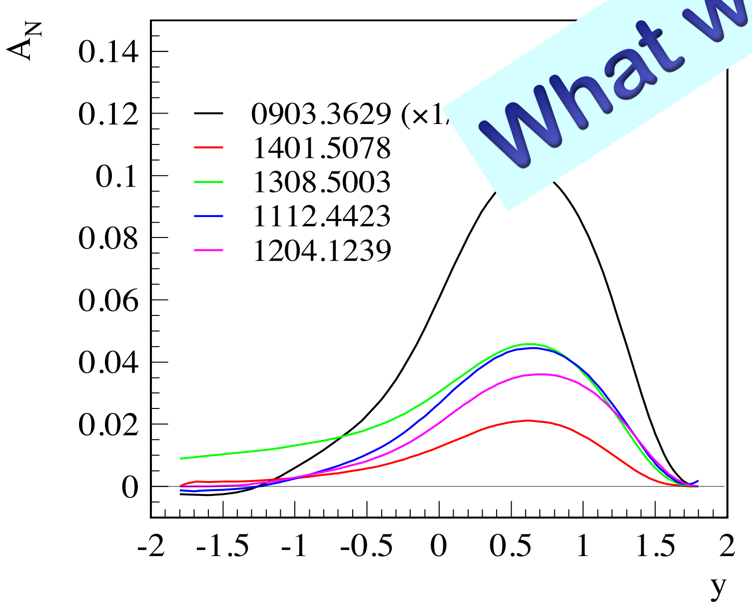
- Fit the same low energy data – Sivers function:

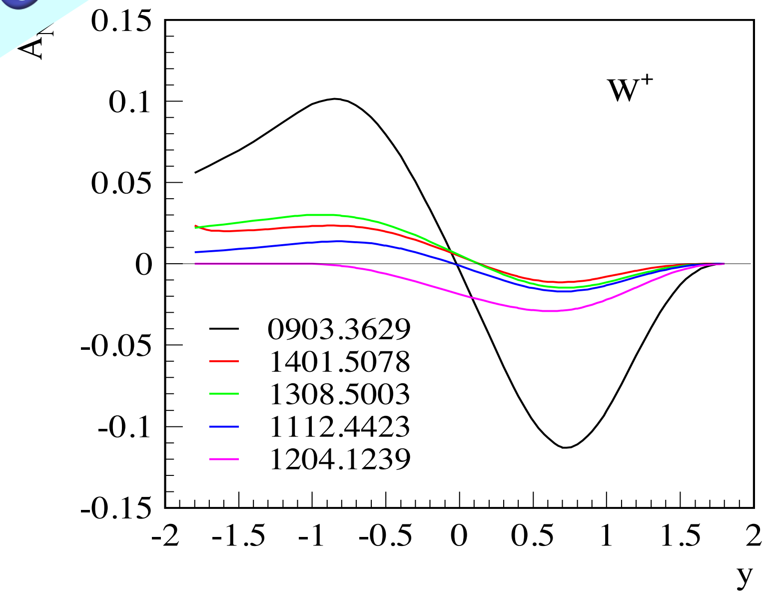




What went wrong?

- Very different "predictions" for ... at higher energy:



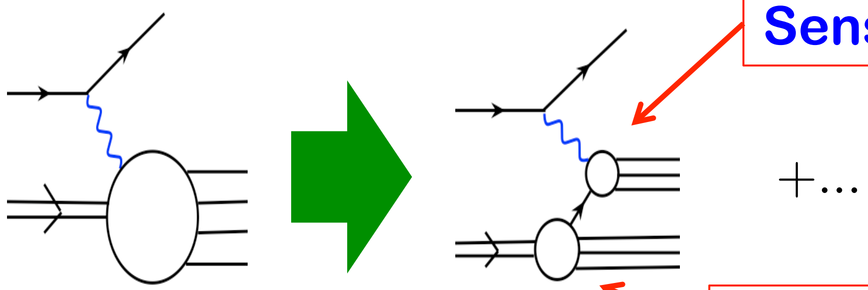


Outline

- Why we need PDFs, TMDs, …?
- Collinear factorization vs. TMD factorization
- Evolution of PDFs vs. evolution of TMDs
- Non-perturbative input for TMD evolution
- Could there be a solution?
- Summary and outlook

QCD factorization is necessary

- Experiments measure hadrons & leptons, neither quarks nor gluons
- Probe of large momentum transfer – sensitive to quarks and gluons:



Sensitive to partonic dynamics

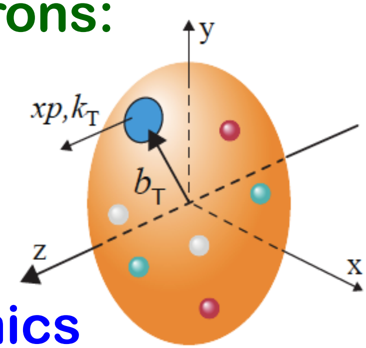
(Diagrams with more active partons from each hadron!)

Connection between hadron and parton

- QCD factorization – connecting quarks & gluons to hadrons:

Hadronic matrix elements of parton fields:

$$\langle p,s|\overline{\psi}(0)\gamma^+\psi(y)|p,s\rangle,\ \ \langle p,s|F^{+\alpha}(0)F^{+\beta}(y)|p,s\rangle(-g_{\alpha\beta})$$



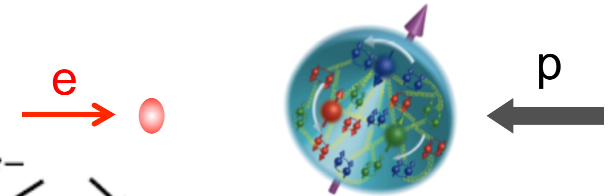
Isolate pQCD calculable short-distance partonic dynamics

No PDFs, No prediction for Higgs cross section, data from the LHC

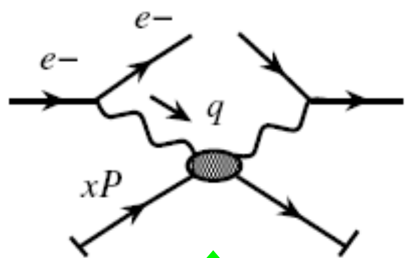
No TMDs, Never "see" the confined motion of quarks and gluons, …

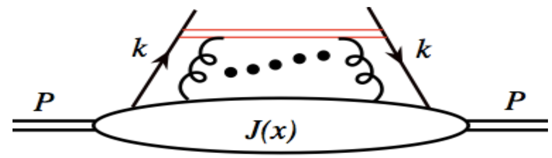
Collinear factorization – single hard scale

❑ **One hadron:**



$Q \gg 1/R \sim \Lambda_{\rm QCD} \sim 1/{\rm fm}$

$\sigma_{\rm tot}^{\rm DIS} \sim$ 

$\otimes$ 

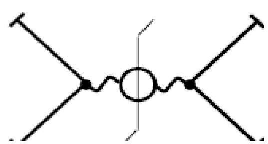
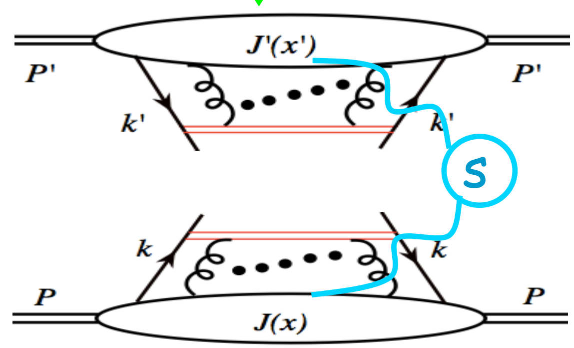
$+ O\left(\frac{1}{QR}\right)$

Hard-part **Probe**

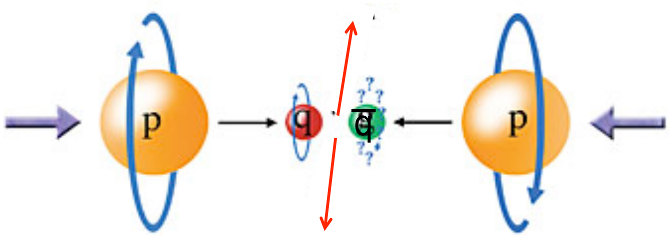
Parton-distribution **Structure**

Power corrections **Approximation**

❑ **Two hadrons:**

$\sigma_{\rm tot}^{\rm DY} \sim$  $\otimes$ 

$+ O\left(\frac{1}{QR}\right)$



Predictive power:
Universal Parton Distributions

Factorization must lead to evolution

- Collinear factorization of DIS structure function:

$$F_2(x_B,Q^2) = \sum_f C_f\left(\frac{x_B}{x},\frac{Q^2}{\mu_F^2},\alpha_s\right)\otimes\varphi_f\left(x,\mu_F^2\right)+O\left(\frac{\Lambda_{\mathrm{QCD}}^2}{Q^2}\right)$$

- Physical cross sections should not depend on factorization scale

$$\mu_F^2\frac{d}{d\mu_F^2}F_2(x_B,Q^2)=0$$

$$\longrightarrow\quad \sum_f\left[\mu_F^2\frac{d}{d\mu_F^2}C_f\left(\frac{x_B}{x},\frac{Q^2}{\mu_F^2},\alpha_s\right)\right]\otimes\varphi_f\left(x,\mu_F^2\right)+\sum_f C_f\left(\frac{x_B}{x},\frac{Q^2}{\mu_F^2},\alpha_s\right)\otimes\mu_F^2\frac{d}{d\mu_F^2}\varphi_f\left(x,\mu_F^2\right)=0$$

- PDFs and coefficient functions share the same logarithms

PDFs: $\log\left(\mu_F^2/\mu_0^2\right)$ or $\log\left(\mu_F^2/\Lambda_{\mathrm{QCD}}^2\right)$

Coefficient functions: $\log\left(Q^2/\mu_F^2\right)$ or $\log\left(Q^2/\mu^2\right)$

→ DGLAP evolution equation:

Perturbative

$$\mu_F^2\frac{\partial}{\partial\mu_F^2}\varphi_i(x,\mu_F^2)=\sum_j P_{i/j}\left(\frac{x}{x'},\alpha_s\right)\otimes\varphi_j(x',\mu_F^2)$$

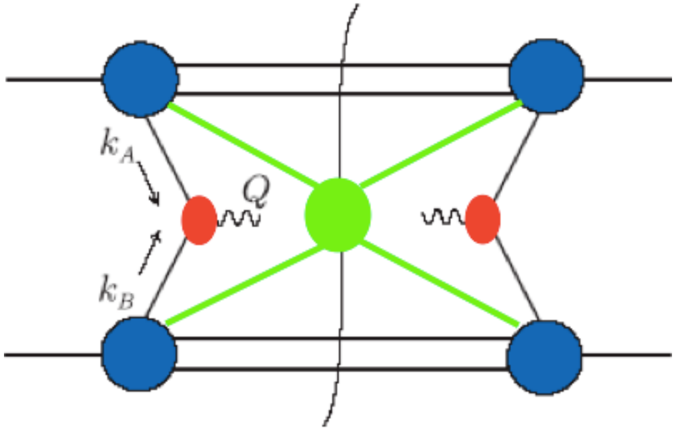
Linear diff-integral Equation:

$\varphi_f(x,\mu_F^2)$ is uniquely fixed, given $\varphi_f(x,\mu_0^2)$

TMD factorization – both hard & soft scale

- **Two hadrons – Drell-Yan:** $Q \gg Q_T \sim \Lambda_{\rm QCD}$

Collins, Soper, Sterman, 1985



$$\frac{d\sigma_{AB}}{dQ^2 dQ_T^2} = \sum_f \int d\xi_a d\xi_b \int \frac{d^2k_{A_T} d^2k_{B_T} d^2k_{s,T}}{(2\pi)^6}$$
$$\times P_{f/A}(\xi_a, k_{A_T}) P_{\bar{f}/B}(\xi_b, k_{B_T}) H_{f\bar{f}}(Q^2) S(k_{s,T})$$
$$\times \delta^2(\vec{Q}_T - \vec{k}_{A_T} - \vec{k}_{B_T} - \vec{k}_{s,T})$$

- **Factorized cross section in "impact parameter b-space":**

$$\delta^2(\vec{Q}_T - \prod_i \vec{k}_{i,T}) = \frac{1}{(2\pi)^2} \int d^2b \, e^{i\vec{b}\cdot\vec{Q}_T} \prod_i e^{-i\vec{b}\cdot\vec{k}_{i,T}}$$

$$\frac{d\sigma_{AB}(Q,b)}{dQ^2} = \sum_f \int d\xi_a d\xi_b \bar{P}_{f/A}(\xi_a, b, n) \bar{P}_{\bar{f}/B}(\xi_b, b, n) H_{f\bar{f}}(Q^2) U(b,n)$$

- **Evolution of TMDs – two equations led to resummation of two log's**

  from the wave function renormalization and the renormalization of the soft factors

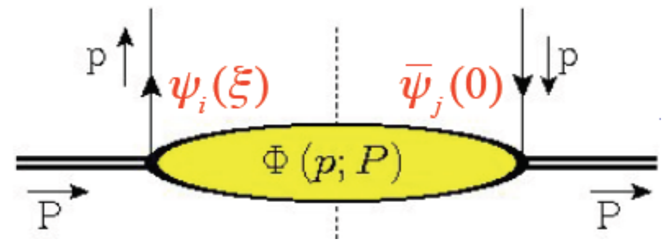
Definitions of TMDs

❑ Non-perturbative definition:

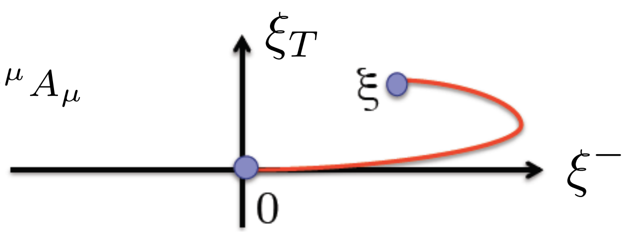
✧ **In terms of matrix elements of parton correlators:**

$$\Phi^{[U]}(x,p_T;n) = \int \frac{d\xi^- d^2\xi_T}{(2\pi)^3}\, e^{i\,p\cdot\xi}\, \langle P,S|\overline{\psi}(0)U(0,\xi)\psi(\xi)|P,S\rangle_{\xi^+=0}$$

✧ **Depends on the choice of the gauge link:**



$$U(0,\xi) = e^{-ig\int_0^\xi ds^\mu A_\mu}$$



✧ **Decomposes into a list of TMDs:**

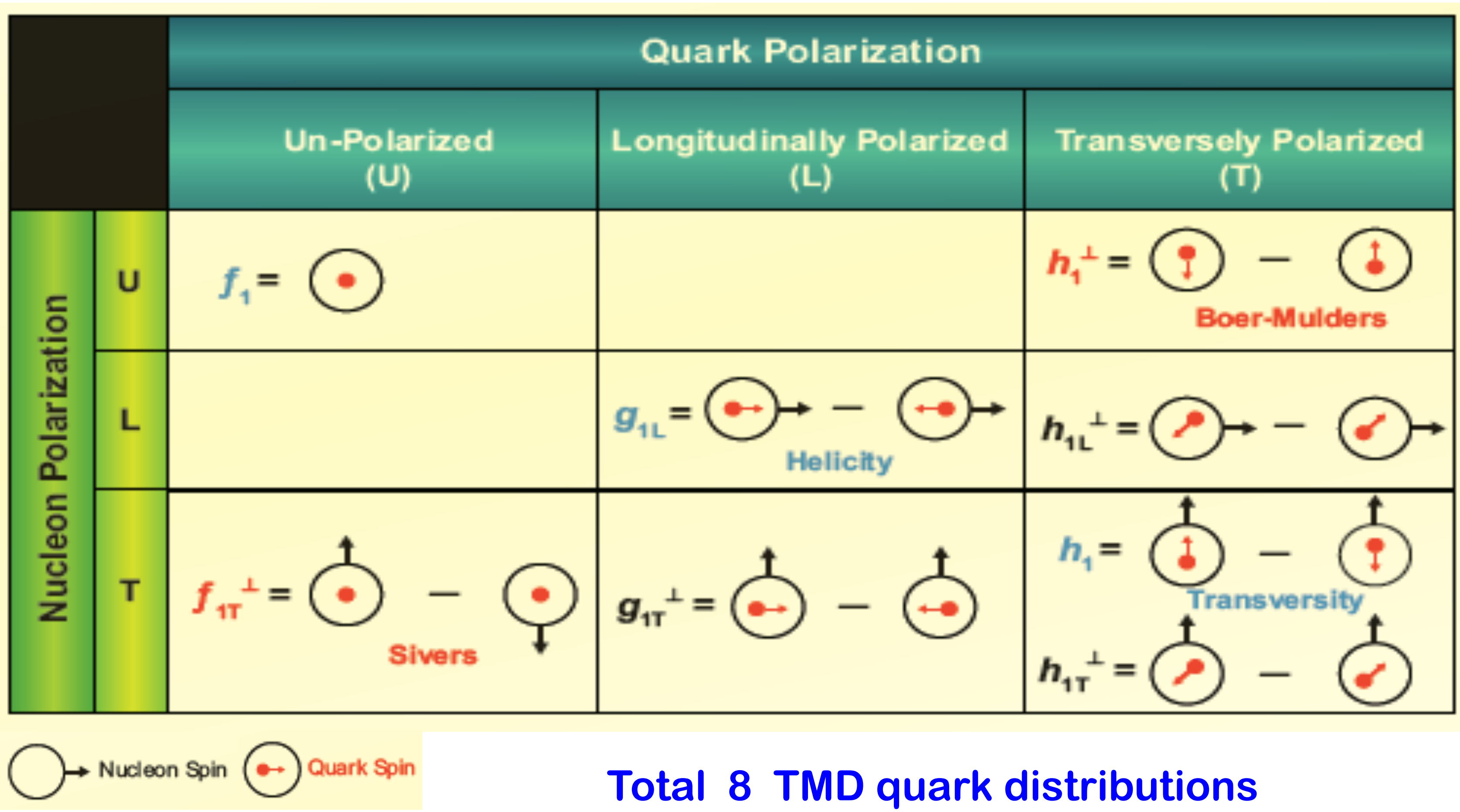
$$\Phi^{[U]}(x,p_T;n) = \Big\{ f_1^{[U]}(x,p_T^2) - f_{1T}^{\perp[U]}(x,p_T^2)\,\frac{\epsilon_T^{p_T S_T}}{M} + g_{1s}^{[U]}(x,p_T)\gamma_5 + h_{1T}^{[U]}(x,p_T^2)\,\gamma_5\,\not{S}_T + h_{1s}^{\perp[U]}(x,p_T)\,\frac{\gamma_5\,\not{p}_T}{M} + ih_1^{\perp[U]}(x,p_T^2)\,\frac{\not{p}_T}{M}\Big\}\frac{\not{P}}{2},$$

✧ **8 TMDs for quark at the leading power (similar to gluon)**

Physical interpretation of TMDs

❑ Quark TMDs:

quantum correlations between hadron and quark spin states



Total 8 TMD quark distributions

Evolution equations for TMDs

J.C. Collins, in his book on QCD

- TMDs in the b-space:

$$\tilde{F}_{f/P^\uparrow}(x, \mathrm{b}_\mathrm{T}, S; \mu; \zeta_F) = \tilde{F}^{\mathrm{unsub}}_{f/P^\uparrow}(x, \mathrm{b}_\mathrm{T}, S; \mu; y_P - (-\infty)) \sqrt{\frac{\tilde{S}_{(0)}(\mathrm{b}_\mathrm{T}; +\infty, y_s)}{\tilde{S}_{(0)}(\mathrm{b}_\mathrm{T}; +\infty, -\infty)\tilde{S}_{(0)}(\mathrm{b}_\mathrm{T}; y_s, -\infty)}} Z_F\, Z_2$$

- Collins-Soper equation:

Renormalization of the soft-factor

$$\frac{\partial \tilde{F}_{f/P^\uparrow}(x, \mathrm{b}_\mathrm{T}, S; \mu; \zeta_F)}{\partial \ln \sqrt{\zeta_F}} = \tilde{K}(b_T; \mu)\tilde{F}_{f/P^\uparrow}(x, \mathrm{b}_\mathrm{T}, S; \mu; \zeta_F)$$

$$\zeta_F = M_P^2 x^2 e^{2(y_P - y_s)}$$

$$\tilde{K}(b_T; \mu) = \frac{1}{2}\frac{\partial}{\partial y_s} \ln\left(\frac{\tilde{S}(b_T; y_s, -\infty)}{\tilde{S}(b_T; +\infty, y_s)}\right)$$

Introduced to regulate the rapidity divergence of TMDs

- RG equations:

Wave function Renormalization

$$\frac{d\tilde{K}(b_T; \mu)}{d \ln \mu} = -\gamma_K(g(\mu))$$

Evolution equations are only valid when $b_T \ll 1/\Lambda_{QCD}$!

$$\frac{d\tilde{F}_{f/P^\uparrow}(x, \mathrm{b}_\mathrm{T}, S; \mu; \zeta_F)}{d \ln \mu} = \gamma_F(g(\mu); \zeta_F/\mu^2)\tilde{F}_{f/P^\uparrow}(x, \mathrm{b}_\mathrm{T}, S; \mu; \zeta_F).$$

- Momentum space TMDs:

Need information at large b_T for all scale μ !

$$F_{f/P^\uparrow}(x, \mathrm{k}_\mathrm{T}, S; \mu, \zeta_F) = \frac{1}{(2\pi)^2}\int d^2\mathrm{b}_T\, e^{i\mathrm{k}_T\cdot \mathrm{b}_T}\, \tilde{F}_{f/P^\uparrow}(x, \mathrm{b}_\mathrm{T}, S; \mu, \zeta_F)$$

Evolution equations for Sivers function

Aybat, Collins, Qiu, Rogers, 2011

- **Sivers function:**

$$F_{f/P^\uparrow}(x,k_T,S;\mu,\zeta_F) = F_{f/P}(x,k_T;\mu,\zeta_F) - F_{1T}^{\perp f}(x,k_T;\mu,\zeta_F)\frac{\epsilon_{ij}k_T^i S^j}{M_p}$$

- **Collins-Soper equation:**

Its derivative obeys the CS equation

$$\frac{\partial \ln \tilde{F}_{1T}^{\prime\perp f}(x,b_T;\mu,\zeta_F)}{\partial \ln \sqrt{\zeta_F}} = \tilde{K}(b_T;\mu)$$

$$\tilde{F}_{1T}^{\prime\perp f}(x,b_T;\mu,\zeta_F) \equiv \frac{\partial \tilde{F}_{1T}^{\perp f}(x,b_T;\mu,\zeta_F)}{\partial b_T}$$

- **RG equations:**

$$\frac{d\tilde{F}_{1T}^{\prime\perp f}(x,b_T;\mu,\zeta_F)}{d\ln\mu} = \gamma_F(g(\mu);\zeta_F/\mu^2)\tilde{F}_{1T}^{\prime\perp f}(x,b_T;\mu,\zeta_F)$$

$$\frac{d\tilde{K}(b_T;\mu)}{d\ln\mu} = -\gamma_K(g(\mu)) \quad \Longrightarrow \quad \frac{\partial\gamma_F(g(\mu);\zeta_F/\mu^2)}{\partial\ln\sqrt{\zeta_F}} = -\gamma_K(g(\mu)),$$

- **Sivers function in momentum space:**

$$F_{1T}^{\perp f}(x,k_T;\mu,\zeta_F) = \frac{-1}{2\pi k_T}\int_0^\infty db_T\, b_T J_1(k_T b_T)\tilde{F}_{1T}^{\prime\perp f}(x,b_T;\mu,\zeta_F)$$

JI, Ma, Yuan, 2004
Idilbi, et al, 2004,
Boer, 2001, 2009,
Kang, Xiao, Yuan, 2011
Aybat, Prokudin, Rogers, 2012
Idilbi, et al, 2012,
Sun, Yuan 2013, …

Extrapolation to large b_T

- **CSS b*-prescription:**

Aybat and Rogers, arXiv:1101.5057
Collins and Rogers, arXiv:1412.3820

$$\tilde{F}_{f/P}(x, \mathrm{b}_T; Q, Q^2) = \overbrace{\sum_j \int_x^1 \frac{d\hat{x}}{\hat{x}} \tilde{C}_{f/j}(x/\hat{x}, b_*; \mu_b^2, \mu_b, g(\mu_b)) f_{j/P}(\hat{x}, \mu_b)}^{\mathrm{AA}}$$
$$\times \overbrace{\times \exp\left\{\ln\frac{Q}{\mu_b}\tilde{K}(b_*;\mu_b) + \int_{\mu_b}^{Q}\frac{d\mu'}{\mu'}\left[\gamma_F(g(\mu');1) - \ln\frac{Q}{\mu'}\gamma_K(g(\mu'))\right]\right\}}^{\mathrm{BB}}$$
$$\times \overbrace{\exp\left\{g_{f/P}(x, b_T) + g_K(b_T)\ln\frac{Q}{Q_0}\right\}}^{\mathrm{CC}}$$

Nonperturbative "form factor"

$$b_* = \frac{b_\mathrm{T}}{\sqrt{1 + b_\mathrm{T}^2/b_\mathrm{max}^2}} \qquad \text{with } b_\mathrm{max} \sim 1/2\ \mathrm{GeV}^{-1}$$

- **Nonperturbative fitting functions**

Various fits correspond to different choices for $g_{f/P}(x, b_T)$ and $g_K(b_T)$

e.g.

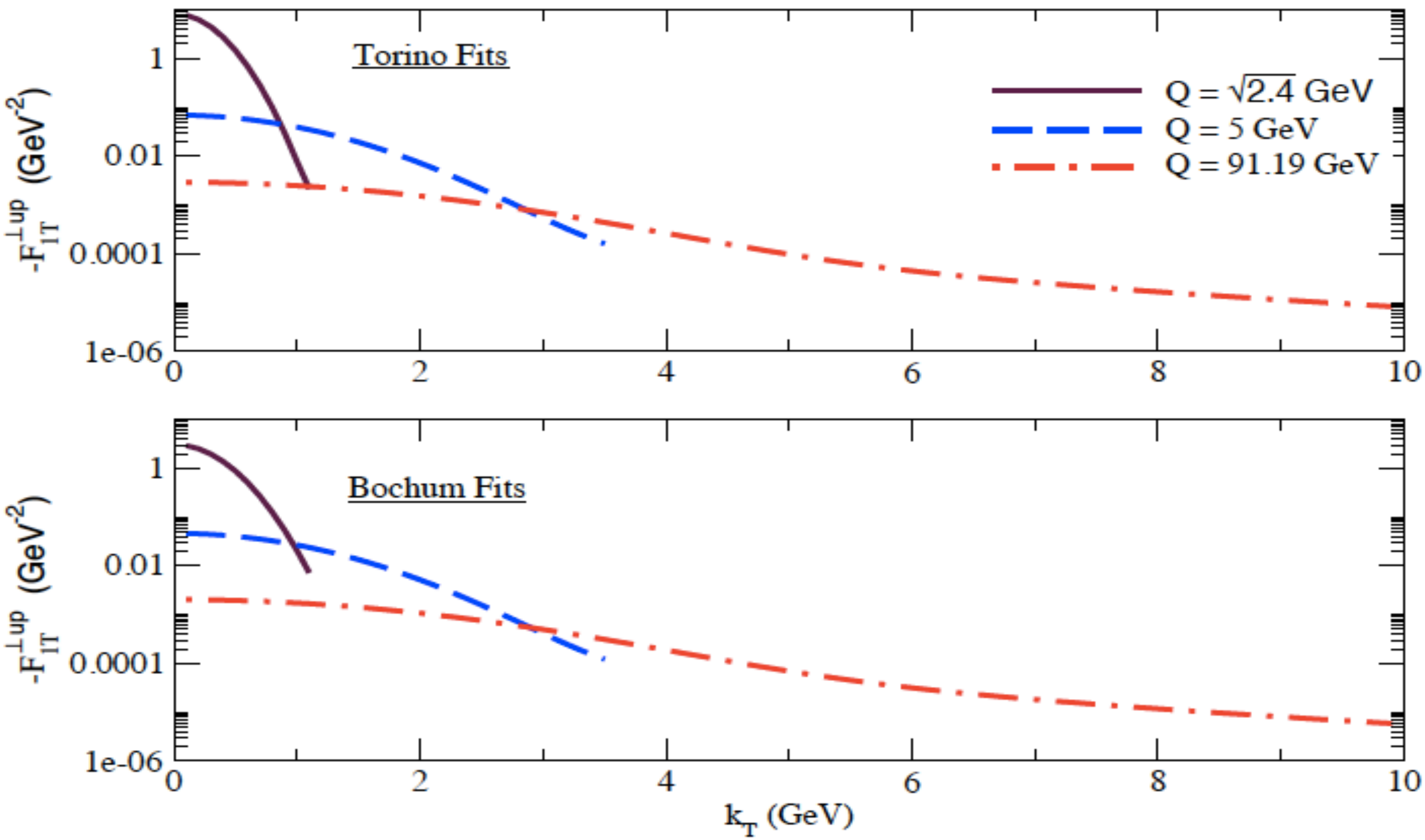
$$g_{f/P}(x, b_T) + g_K(b_T)\ln\frac{Q}{Q_0} \equiv -\left[g_1 + g_2\ln\frac{Q}{2Q_0} + g_1 g_3 \ln(10x)\right] b_T^2$$

Different choice of g2 & b_* could lead to different over all Q-dependence!

Evolution of Sivers function

Aybat, Collins, Qiu, Rogers, 2011

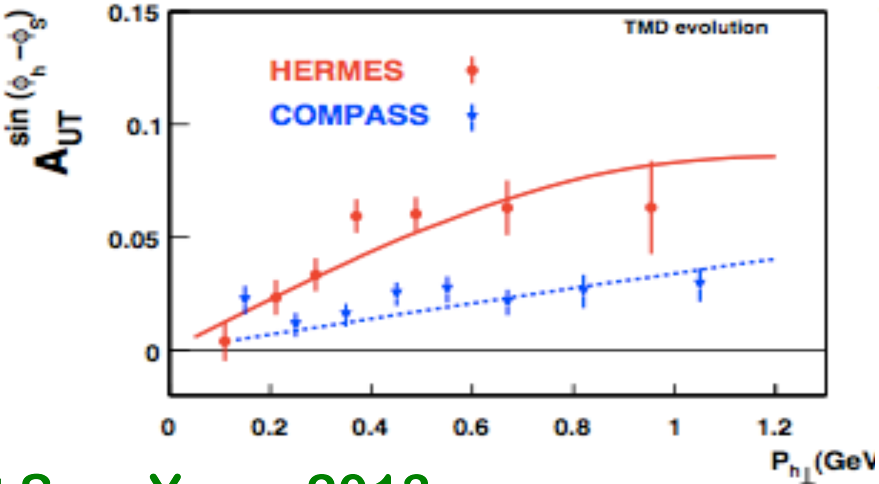
- Up quark Sivers function:

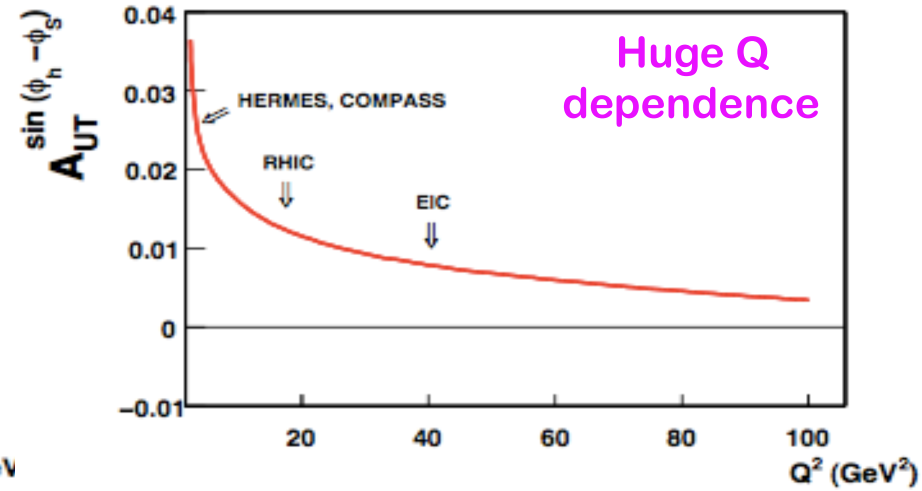


Very significant growth in the width of transverse momentum

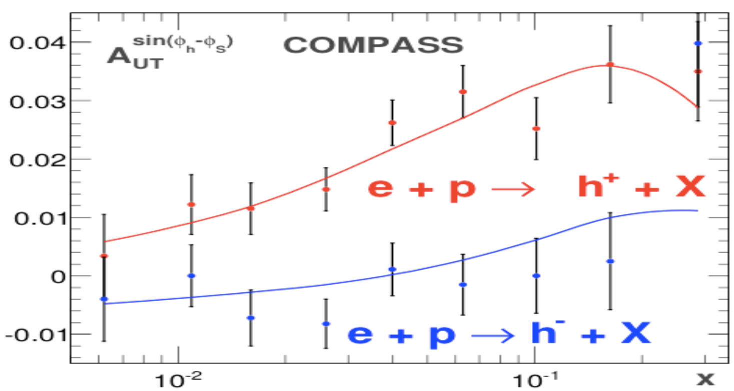
Different fits – different Q-dependence

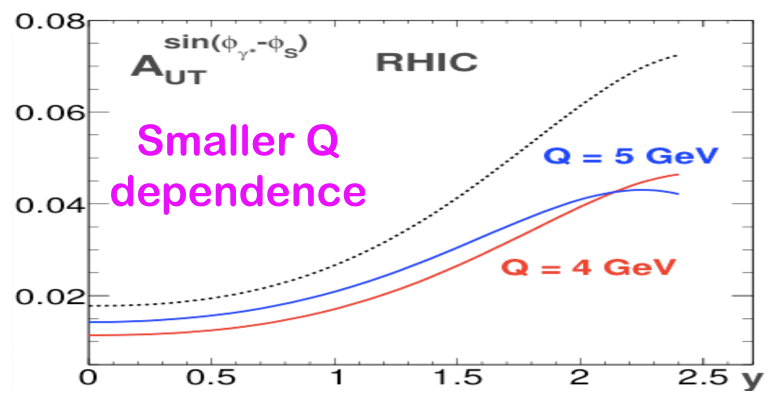
- Aybat, Prokudin, Rogers, 2012:





- Sun, Yuan, 2013:





No disagreement on evolution equations!

Issues: Extrapolation to non-perturbative large b-region
Choice of the Q-dependent "form factor"

What happened?

- Sivers function:

Differ from PDFs!

$$F_{1T}^{\perp f}(x,k_T;\mu,\zeta_F) = \frac{-1}{2\pi k_T}\int_0^\infty db_T\, b_T\, J_1(k_T b_T)\tilde{F}_{1T}^{\prime\perp f}(x,b_T;\mu,\zeta_F)$$

Need non-perturbative large b_T information for any value of Q! *$Q=\mu$*

- What is the “correct” Q-dependence of the large b_T tail?

$$\tilde{F}_{f/P}(x,\mathrm{b}_T;Q,Q^2) = \overbrace{\sum_j\int_x^1\frac{d\hat{x}}{\hat{x}}\tilde{C}_{f/j}(x/\hat{x},b_*;\mu_b^2,\mu_b,g(\mu_b))f_{j/P}(\hat{x},\mu_b)}^{\mathrm{AA}}$$

$$\times\ \overbrace{\times\exp\left\{\ln\frac{Q}{\mu_b}\tilde{K}(b_*;\mu_b)+\int_{\mu_b}^{Q}\frac{d\mu'}{\mu'}\left[\gamma_F(g(\mu');1)-\ln\frac{Q}{\mu'}\gamma_K(g(\mu'))\right]\right\}}^{\mathrm{BB}}$$

$$\times\ \overbrace{\exp\left\{g_{f/P}(x,b_T)+g_K(b_T)\ln\frac{Q}{Q_0}\right\}}^{\mathrm{CC}}$$

Nonperturbative “form factor”

$$g_{f/P}(x,b_T)+g_K(b_T)\ln\frac{Q}{Q_0}\equiv-\left[g_1+g_2\ln\frac{Q}{2Q_0}+g_1g_3\ln(10x)\right]b_T^2$$

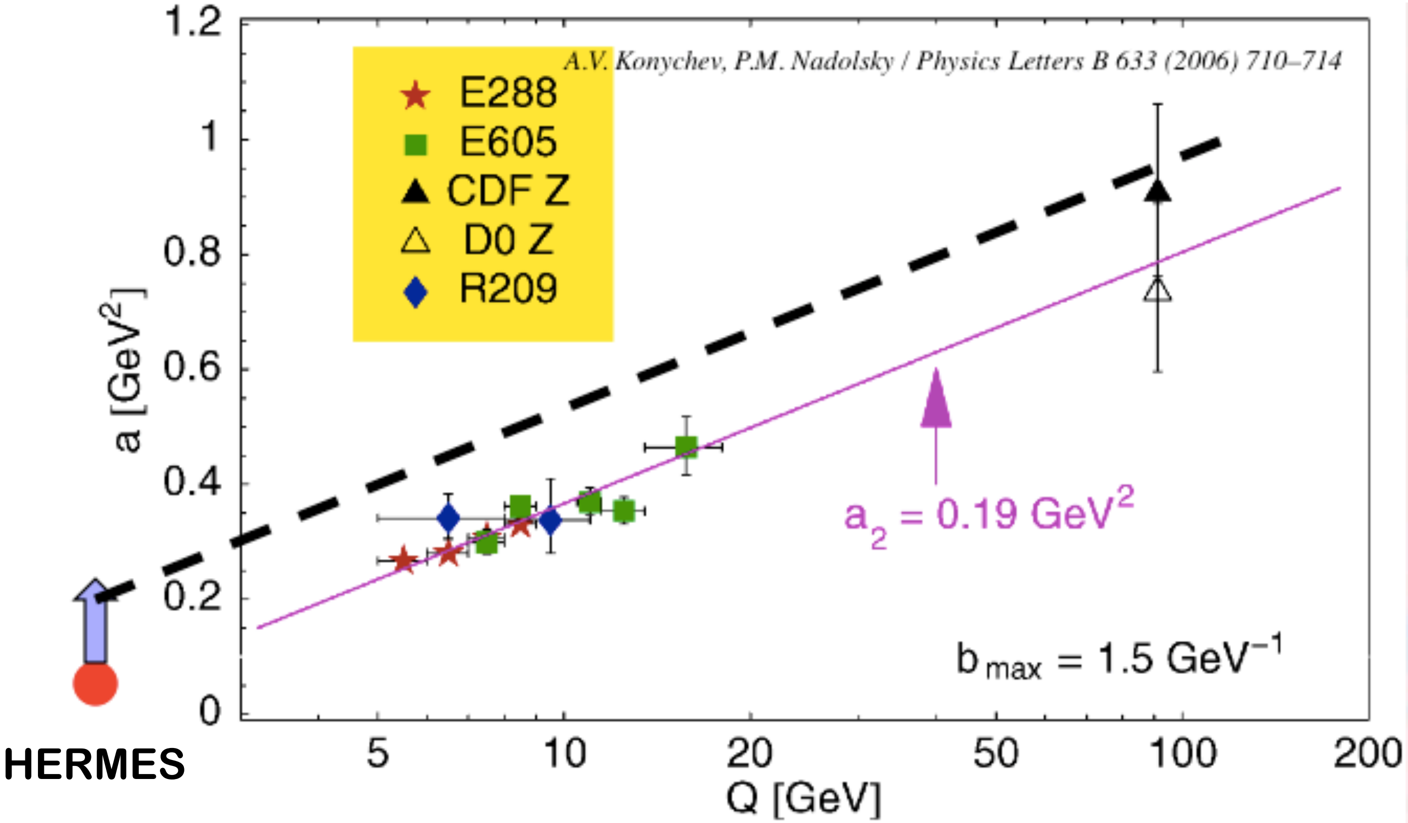
Is the log(Q) dependence sufficient? *Choice of g_2 & b_* affects Q-dep.*

The “form factor” and b_* change perturbative results at small b_T!

Q-dependence of the “form” factor

- Q-dependence of the “form factor” :

Konychev, Nadolsky, 2006



$$\mathcal{F}^{\rm NP}(b,Q) = a(Q^2)\, b^2$$

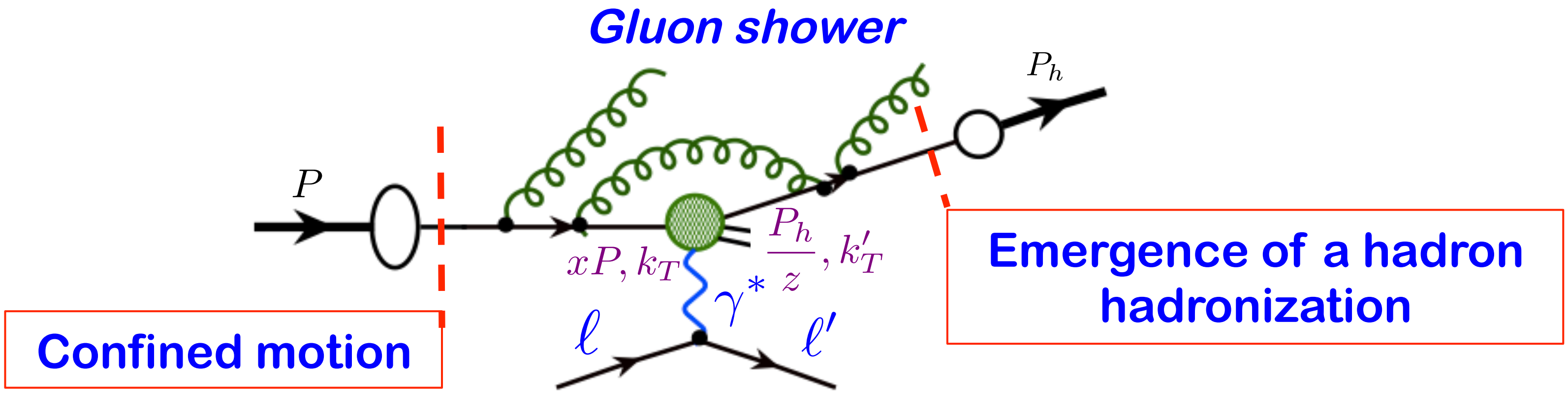
At Q ~ 1 GeV, ln(Q/Q_0) term may not be the dominant one!

$$\mathcal{F}^{\rm NP} \approx b^2(a_1 + a_2 \ln(Q/Q_0) + a_3 \ln(x_A x_B) + \ldots) + \ldots$$

Power correction? **$(Q_0/Q)^n$-term?** ***Better fits for HERMES data?***

Parton k_T at the hard collision

- Sources of parton k_T at the hard collision:



- Large k_T generated by the shower (caused by the collision):
  - ✧ Q^2-dependence – linear evolution equation of TMDs in b-space
  - ✧ The evolution kernels are perturbative at small b, but, not large b
  - ➡ The nonperturbative inputs at large b could impact TMDs at all Q^2

- Challenge: to extract the "true" parton's confined motion:
  - ✧ Separation of perturbative shower contribution from nonperturbative hadron structure – not as simple as PDFs

What controls the b-space distribution?

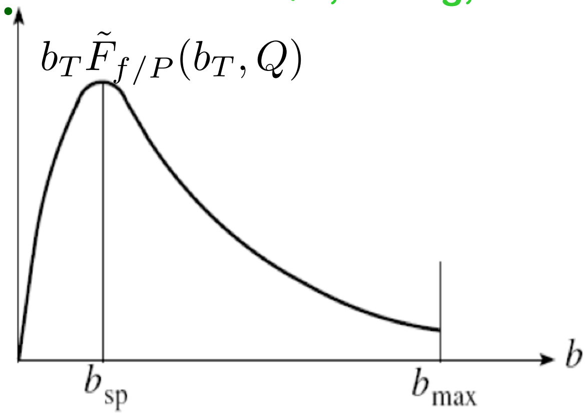
❑ **Features of perturbative calculation at small-b:**

Qiu, Zhang, 2001

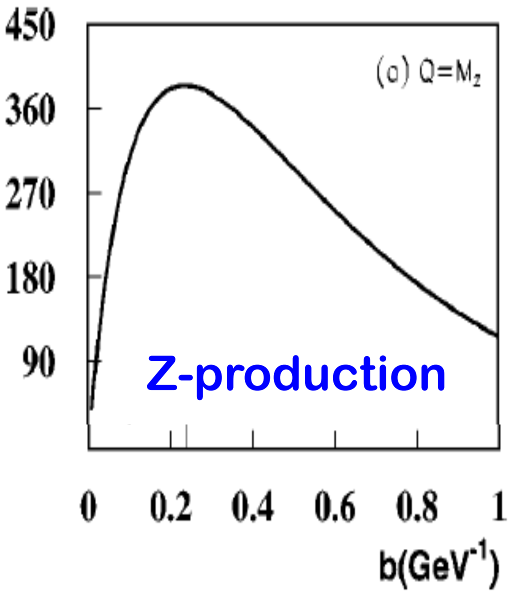
– Sudakov form factor $\rightarrow\ b_{sp} \propto (\frac{\Lambda_{QCD}}{Q})^\lambda, \lambda \sim 0.5$

– evolution of $f_{a/A}$ and $D_{c\rightarrow h}$ also moves b_{sp}

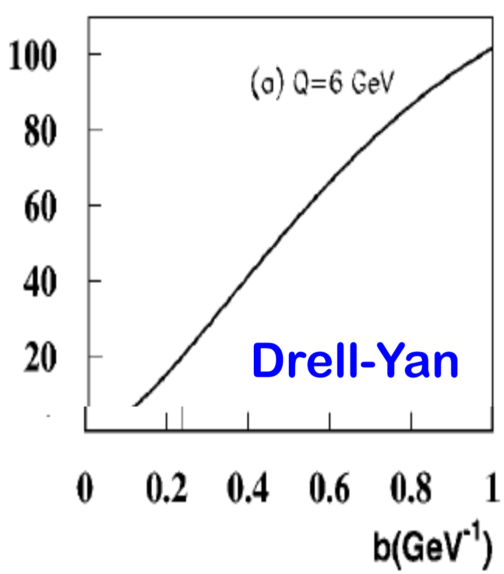
smaller $\xi \Rightarrow \mu\frac{\partial}{\partial\mu} f_{a/A}(\xi) > 0 \Rightarrow$ lower b_{sp}



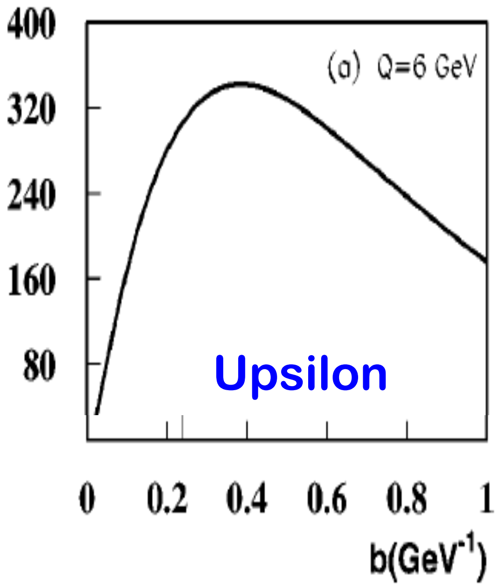
❑ **b-space distribution, and its Q and √s dependence:**



$\sqrt{s} = 1.8$ TeV



$\sqrt{s} = 27.4$GeV



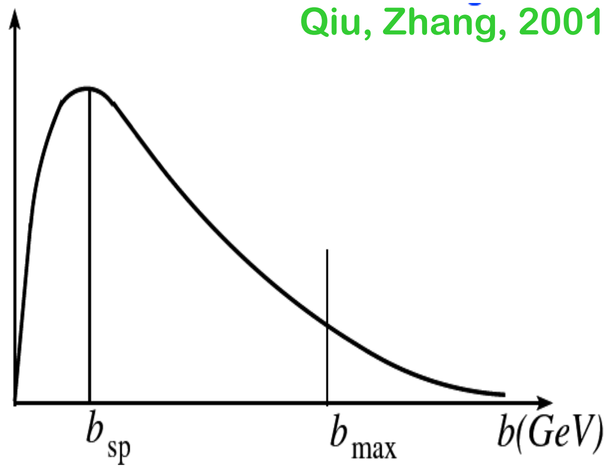
$\sqrt{s} = 1.8$ TeV

Small contribution from large-b_T

Qiu, Zhang, 2001

- Preserve calculated result at small b_T:

$$\frac{d\sigma^{\text{resum}}_{AB\to Z}}{dq_T^2} \propto \int_0^\infty db\, J_0(q_T b)\, b\, W(b,Q)$$



$$W = \begin{cases} W^{\text{pert}}(b,x,z,Q) & b \le b_{max} \\ W^{\text{pert}}(b_{max},x,z,Q)\, F^{NP}(b,Q;b_{max}) & b > b_{max} \end{cases}$$

$$W^{\text{pert}}(b,x,z,Q) = \sum_j e_j^2 \left[f_{a/A} \otimes C^{in}_{a\to j}\right]\left[C^{out}_{j\to c} \otimes D_{b\to h}\right] \times e^{-S(b,Q)}$$

$$F^{NP}_{QZ}(b,Q;b_{max}) = \exp\left\{ -\ln\left(\frac{Q^2 b_{max}^2}{c^2}\right)\left[g_1\left((b^2)^\alpha - (b_{max}^2)^\alpha\right) + g_2\left(b^2 - b_{max}^2\right)\right] - \bar{g}_2\left(b^2 - b_{max}^2\right)\right\}$$

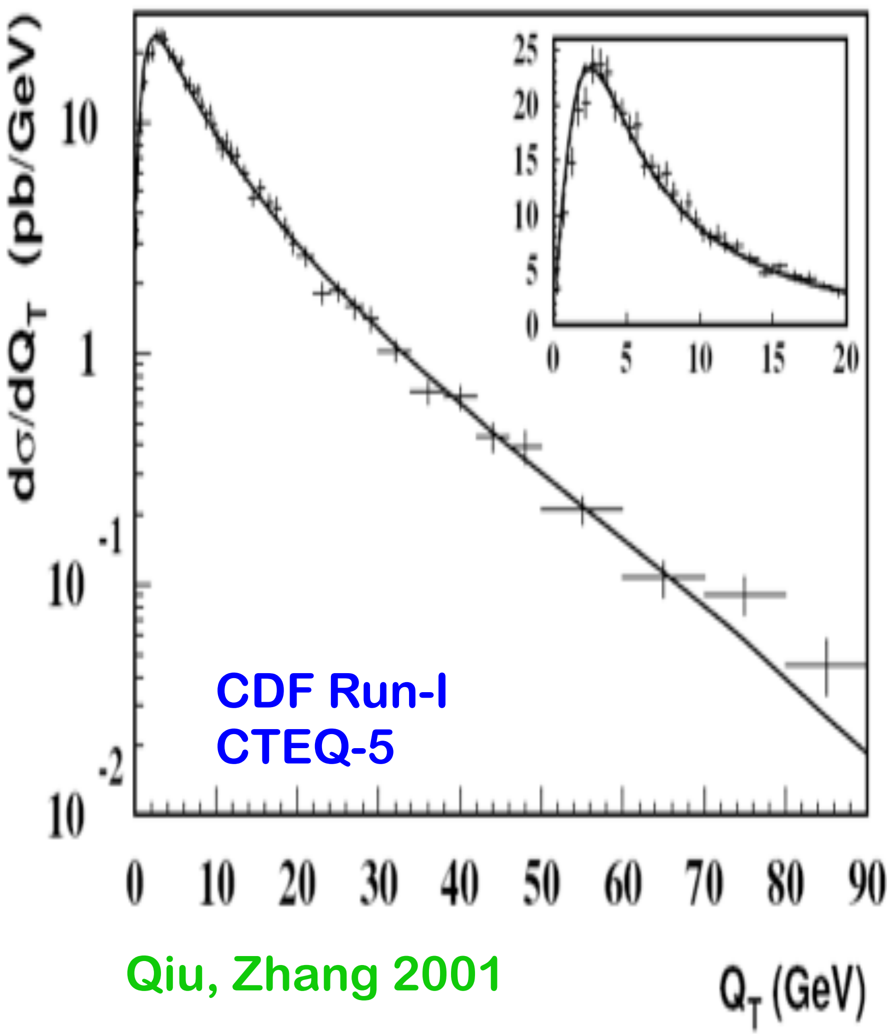
Leading twist

Intrinsic power corrections

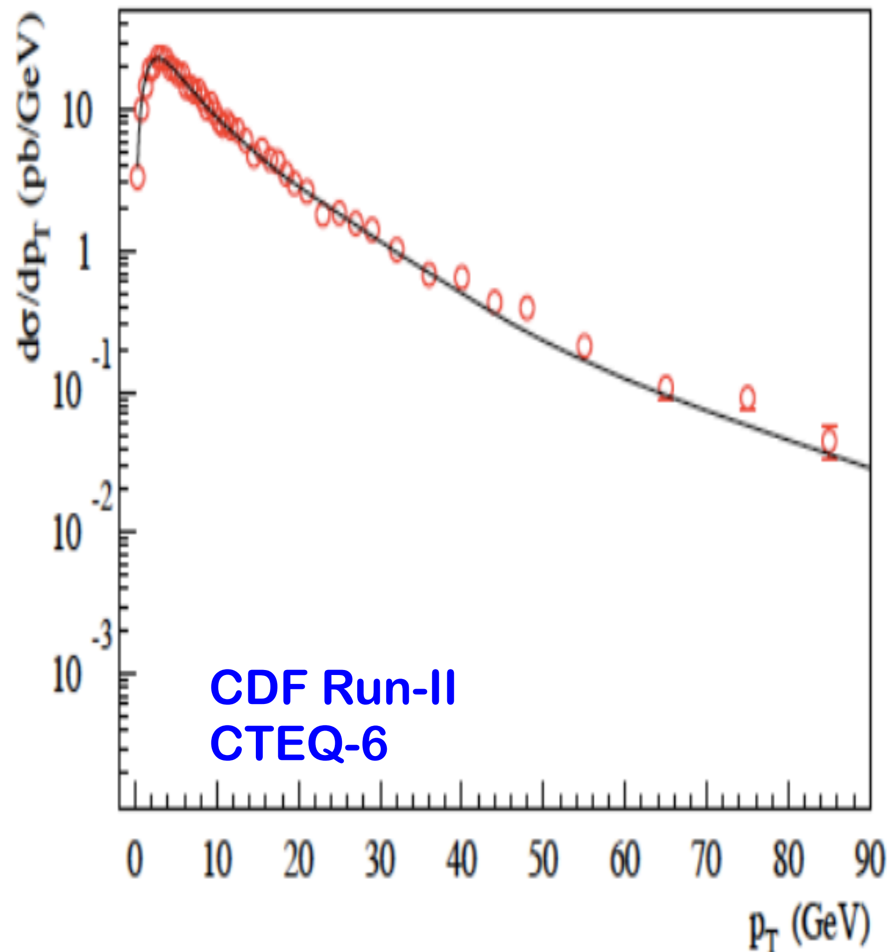
Dynamical power corrections

All parameters, α, g_1, g_2, are fixed by the continuity of the "W" and its derivatives at b_{max} – excellent predictive power for observables with the saddle point at small enough b_{sp}

Phenomenology – Z^0 at Tevatron



Qiu, Zhang 2001

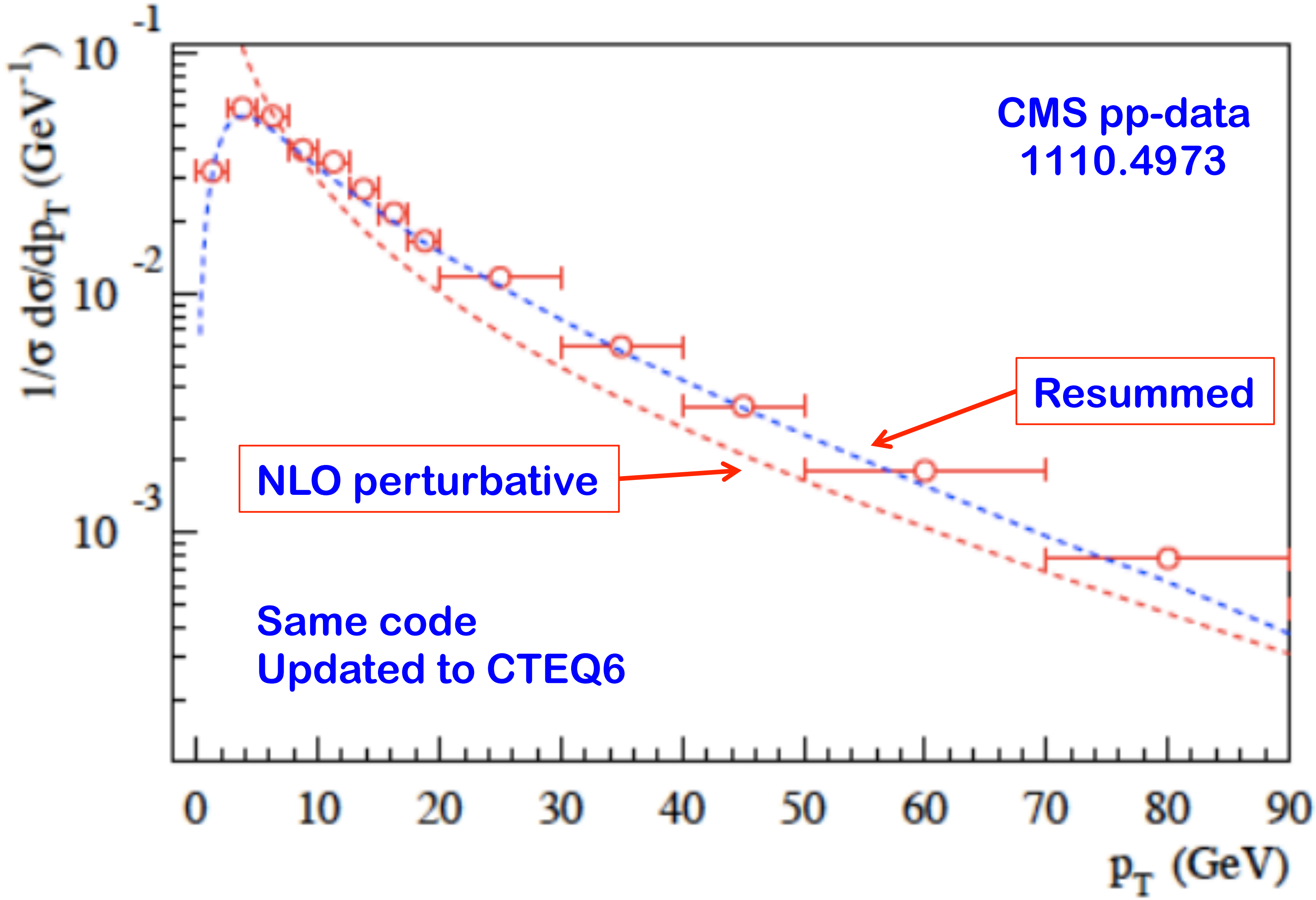


Kang, Qiu 2012

No free fitting parameter!

Phenomenology – Z^0 at the LHC

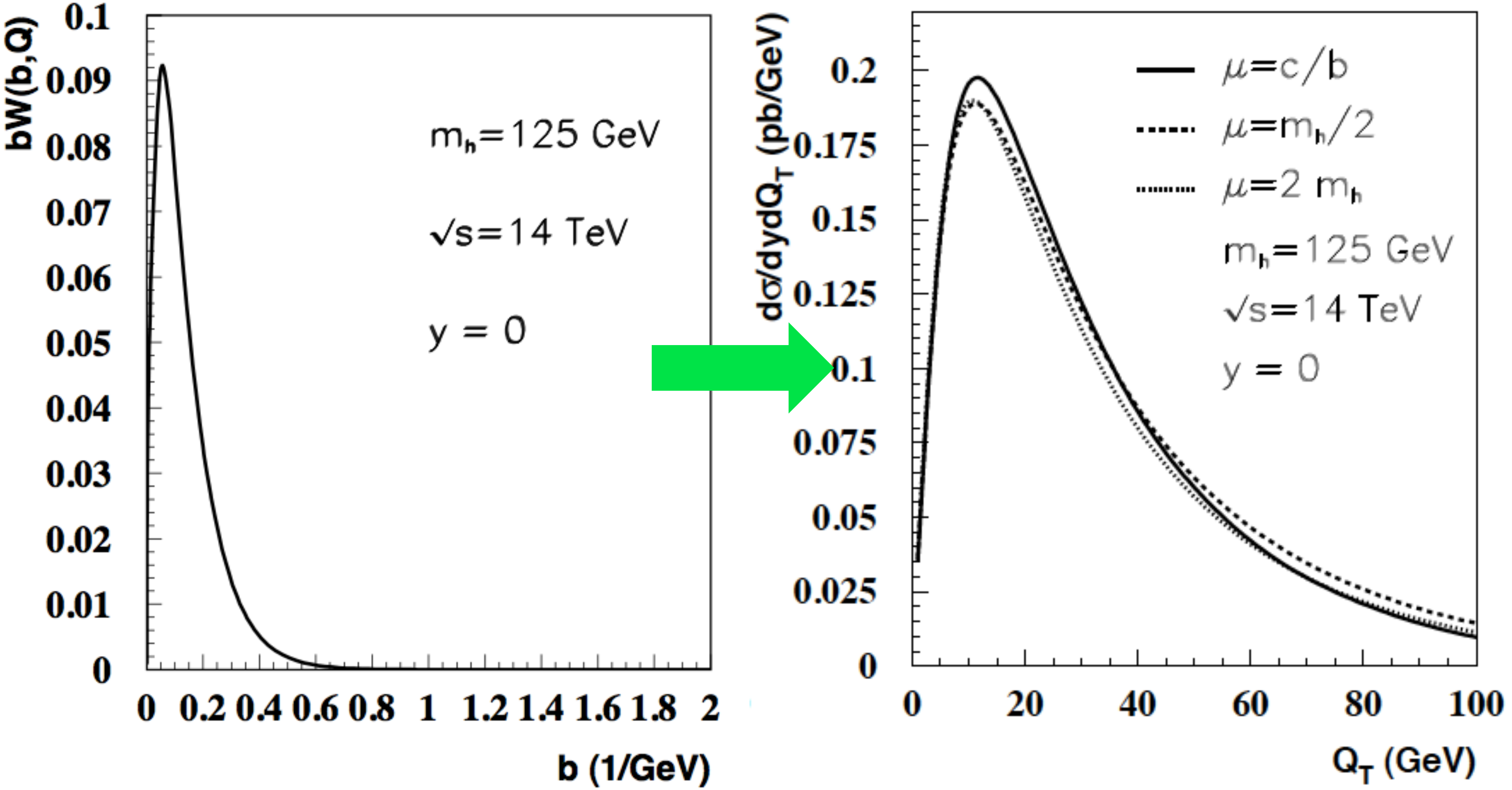
Kang, Qiu, 2012



Effectively no non-perturbative uncertainty!

Phenomenology – Higgs

Berger, Qiu, 2003

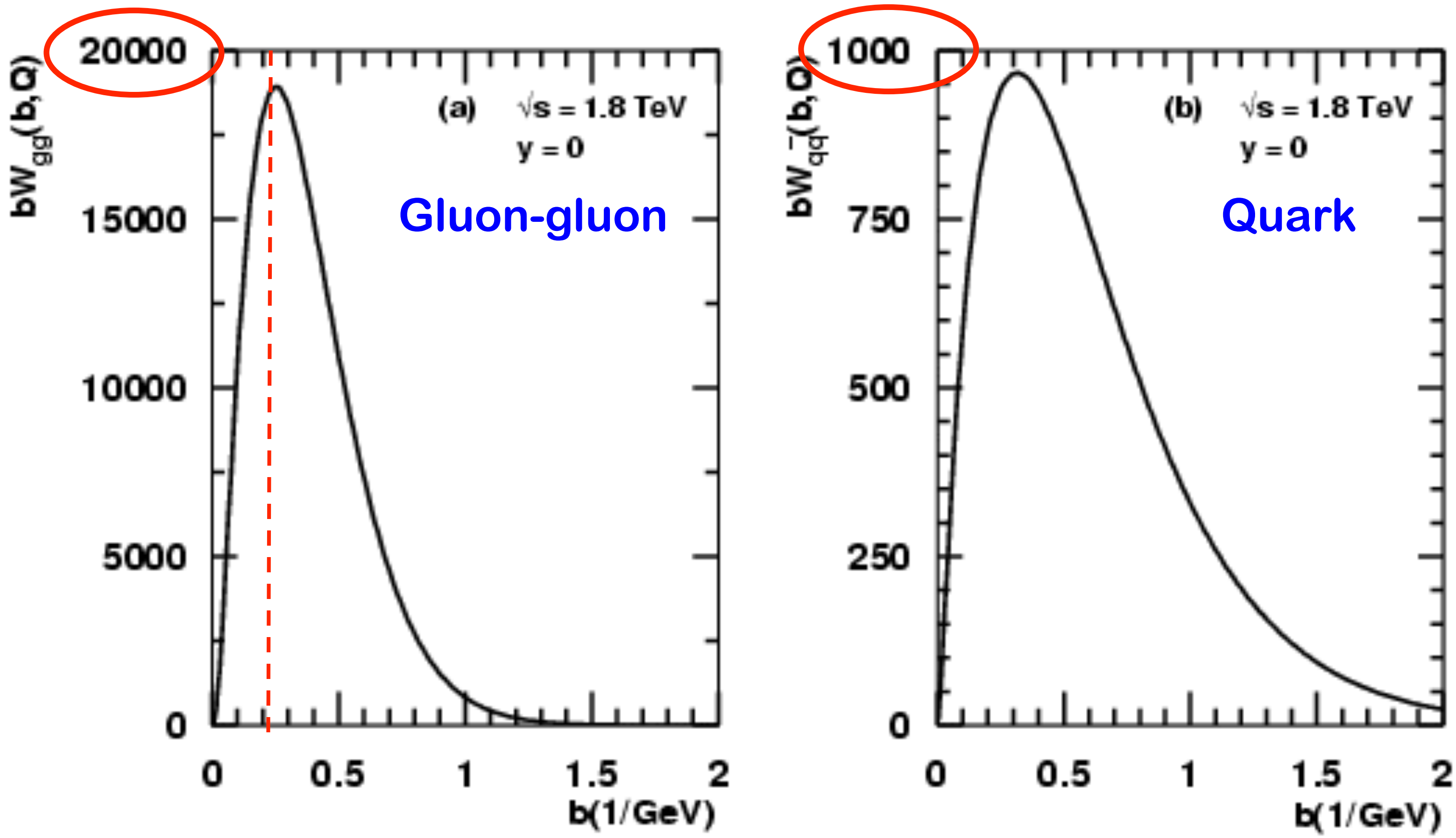


Effectively no non-perturbative uncertainty!

Phenomenology – Upsilon production

Berger, Qiu, Wang, 2005

- Upsilon production (low Q, large phase space):



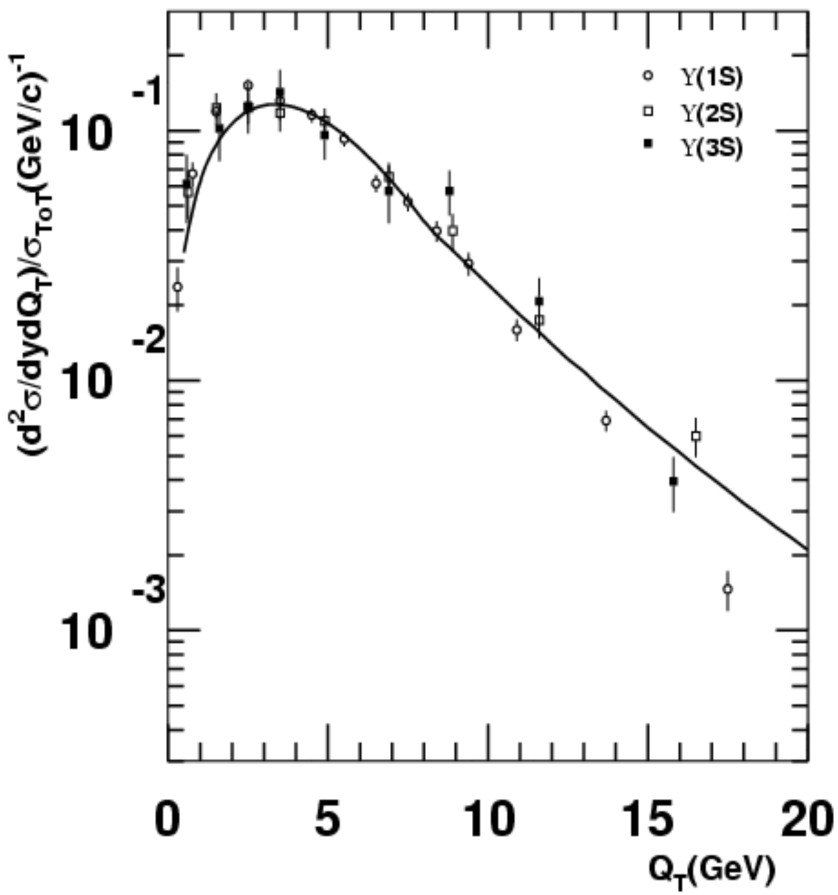
Gluon-gluon dominate the production

Dominated by perturbative contribution even M_Υ~10 GeV

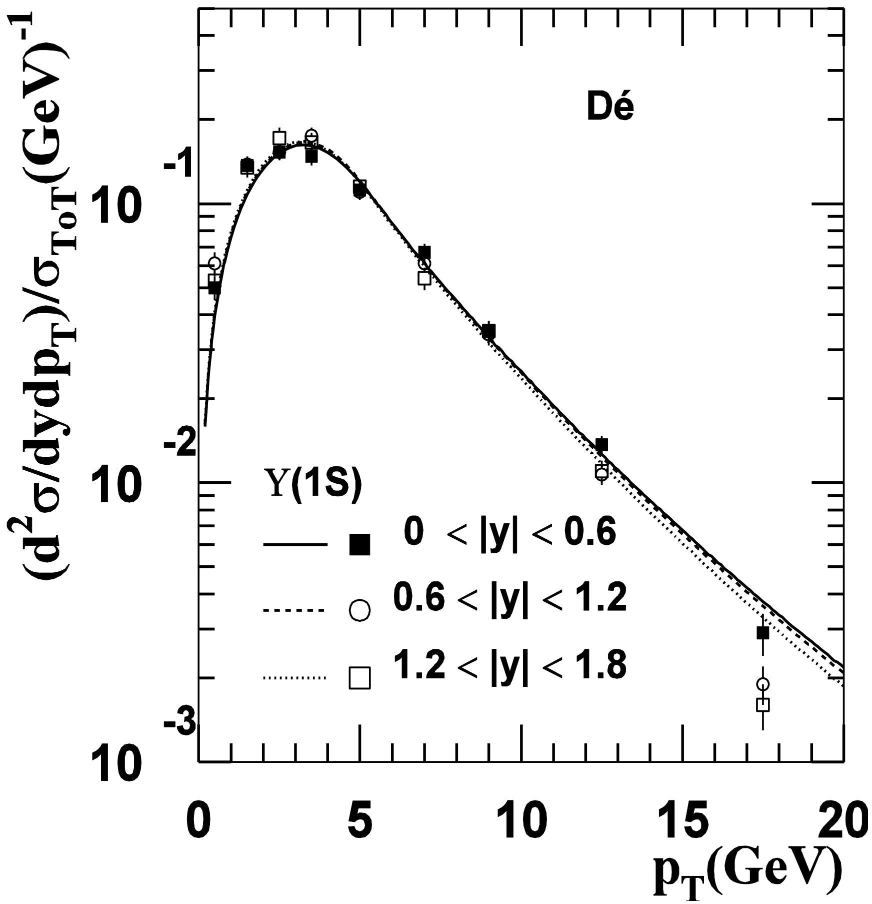
Phenomenology

Berger, Qiu, Wang, 2005

- Prediction vs Tevatron data:



CDF Run-I data



DO Run-II data

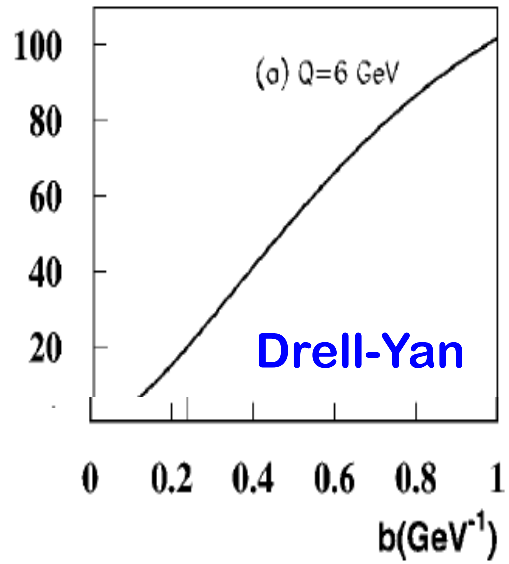
Observables sensitive to the large b_T

- Saddle point is in nonperturbative regime:

Qiu, Zhang, 2001

Low energy Drell-Yan and low energy SIDIS

$$\sqrt{s} = 27.4\text{GeV}$$



b-space distribution is dominated by large b_T region

- Possible solution:

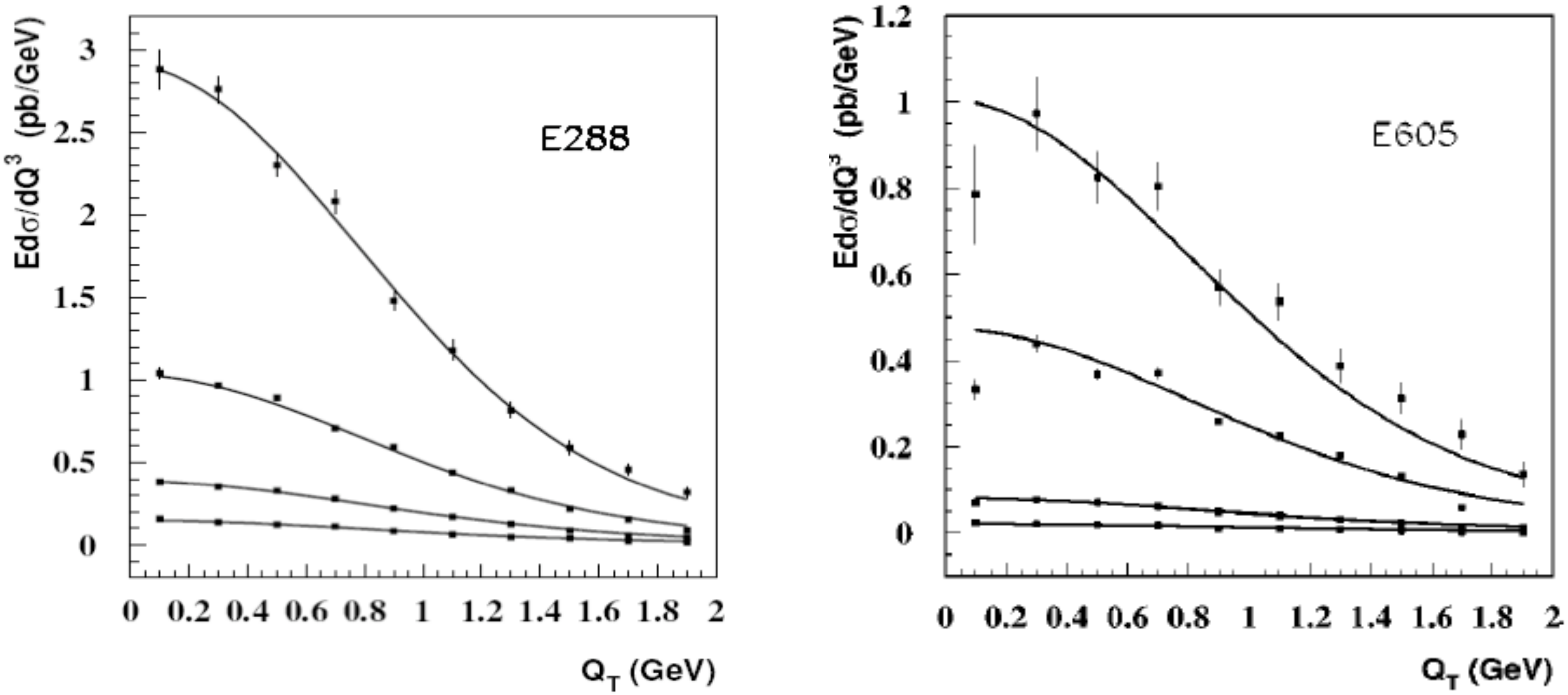
Kang, Qiu in preparation

- Bessel function help suppress the large b_T contribution
- Preserve pQCD calculation at small b_T
- Simple logarithmic Q-dependence of the form factor is not sufficient
- Observation:
  - Large b_T – small k_T – active parton is nearly collinear
  - Develop a better extrapolation by resummation of power corrections

Phenomenology – Drell-Yan

- Leading power correction form is already good:

Qiu, Zhang, 2001



$$F^{NP}_{QZ}(b,Q;b_{max}) = \exp\left\{ -\ln\left(\frac{Q^2 b^2_{max}}{c^2}\right)\left[g_1\left((b^2)^\alpha - (b^2_{max})^\alpha\right) + g_2\left(b^2 - b^2_{max}\right)\right] - \bar{g}_2\left(b^2 - b^2_{max}\right)\right\}$$

Leading twist

Intrinsic power corrections

Dynamical power corrections

Proposal from Collins and Roger

- "Resummed" large b_T behavior:

Collins and Rogers, arXiv:1412.3820

$$\tilde{F}_{f/P}(x, b_T; Q, Q^2) = \overbrace{\sum_j \int_x^1 \frac{d\hat{x}}{\hat{x}} \tilde{C}_{f/j}(x/\hat{x}, b_*; \mu_b^2, \mu_b, g(\mu_b)) f_{j/P}(\hat{x}, \mu_b)}^{\text{AA}}$$

$$\times \overbrace{\times \exp\left\{ \ln\frac{Q}{\mu_b} \tilde{K}(b_*; \mu_b) + \int_{\mu_b}^{Q} \frac{d\mu'}{\mu'} \left[\gamma_F(g(\mu'); 1) - \ln\frac{Q}{\mu'} \gamma_K(g(\mu')) \right] \right\}}^{\text{BB}}$$

$$\times \overbrace{\exp\left\{ g_{f/P}(x, b_T) + g_K(b_T) \ln\frac{Q}{Q_0} \right\}}^{\text{CC}}$$

Nonperturbative "form factor"

$$g_{f/P}(x, b_T) + g_K(b_T) \ln\frac{Q}{Q_0} \equiv -\left[g_1 + g_2 \ln\frac{Q}{2Q_0} + g_1 g_3 \ln(10x) \right] b_T^2$$

$$g_K(b_T; b_{\max}) = g_0(b_{\max}) \left(1 - \exp\left[-\frac{C_F \alpha_s(\mu_{b_*}) b_T^2}{\pi g_0(b_{\max})\, b_{\max}^2} \right] \right)$$

$$g_0(b_{\max}) = g_0(b_{\max,0}) + \frac{2C_F}{\pi} \int_{C_1/b_{\max,0}}^{C_1/b_{\max}} \frac{d\mu'}{\mu'} \alpha_s(\mu')$$

$$\Longrightarrow \frac{C_F}{\pi} \frac{b_T^2}{b_{\max}^2} \alpha_s(\mu_{b_*}) + \mathcal{O}(b_T^4)$$

Summary

- TMDs are NOT direct physical observables
  – could be defined differently

  Relevant definition arises from the approximation used in deriving the factorization!

- The evolution equations of the TMDs are in b-space, and are the consequence of the factorization

- Knowledge of nonperturbative inputs at large b is crucial in determining the TMDs from fitting the data

- The TMD Collaboration – a topical theory collaboration was formed to pull together expertise from theory, lattice and phenomenology to address issues concerning TMDs

Thank you!