On the possibility of multidimensional structure of inhomogeneous chiral phases

~ Toward a search for multidimensional chiral crystals ~

Tong-Gyu Lee (Kyoto Univ.)

w/ N. Yasutake (CIT), T. Maruyama (JAEA), K. Nishiyama and T. Tatsumi (Kyoto Univ.)

J-PARC hadron physics in 2016

IQBRC, Tokai, Ibaraki, Japan, Mar. 2-4, 2016

Nonvanishing baryon density

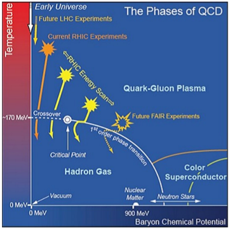
- Dense QCD phase diagram

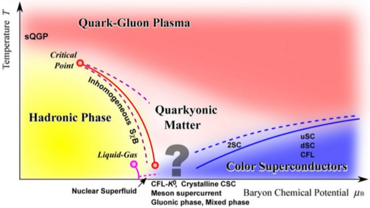
  Data at lower beam energies will be forthcoming at J-PARC, FAIR, Dubna, RHIC(BES-II), ...

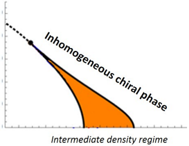
  Finding appropriate observables are needed

  One might expect a transition to exotic phases because of high densities

  Recent theoretical studies predict inhomogeneous phases







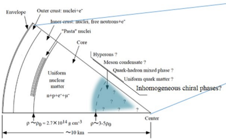
Nonvanishing baryon density

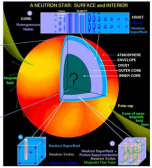
- Compact star physics:

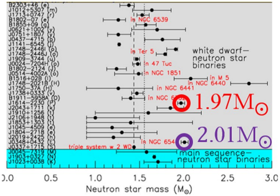
  Inside of compact stellar objects reaches high densities
  $\rightarrow$ might be reasonable to expect a transition to exotic phases

  Discovery of two-solar-mass neutron stars
  $\rightarrow$ EOS should be stiff to support massive stars
  $\rightarrow$ consideration of exotic phases could become important if it has a stiffer EOS
  $\rightarrow$ inhomogeneous phases in QM core might make it possible... 【Carignano-Ferrer-Incera(2015)】

  One expects the development of compact star physics via the observations at ASTRO-H (launched recently), LIGO (detected gravitational waves), etc.





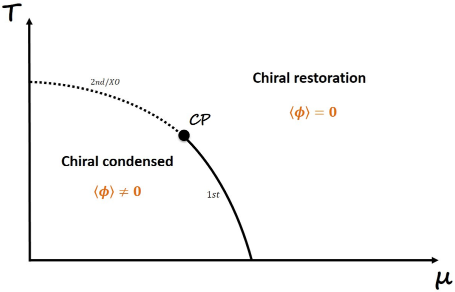


- Inhomogeneous phases are relevant for dense matter

  $\rightarrow$ We here focus on inhomogeneous chiral phases

Inhomogeneous chiral phase

- Conventional phase diagram (focused on a chiral phase transition)

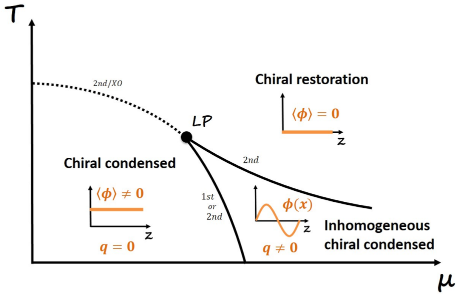


$\Rightarrow$ order parameter is constant in space (homogeneous condensed phase)

$\Rightarrow$ what if a space dependent one is allowed and lowers free energies?

Inhomogeneous chiral phase

- The resulting phase diagram (focused on a chiral transition region)

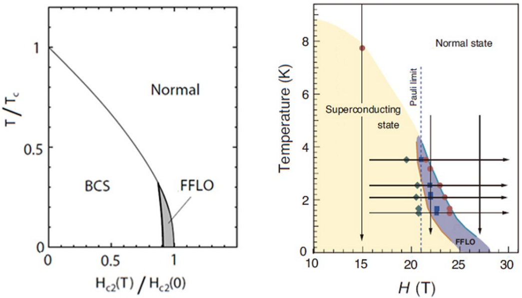


⇒ chiral transition region is extended (where an inhomogeneous chiral phase appears)

⇒ obtained from, e.g., NJL mean-field calculations 【cf. Buballa-Carignano;PPNP(2015)】

Inhomogeneous phases

- Condensed matter physics (interplay of superconductivity and ferromagnetism)

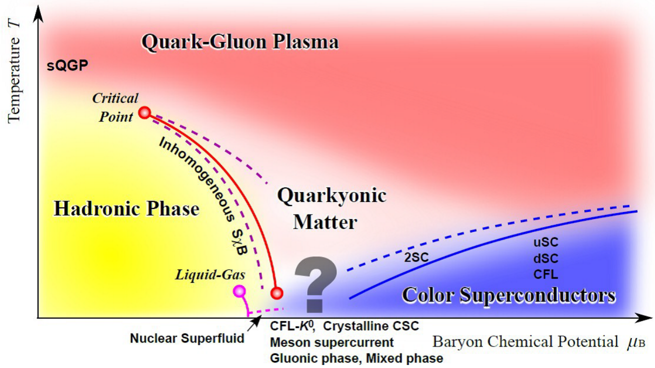


⇒ Well-known inhomogeneous phase includes a FFLO state 【cf. Matsuda-Shimahara(2007)】

⇒ It has been observed a FFLO phase in an organic superconductor 【Mayaffre et al.(2014)】

Inhomogeneous chiral phase

- Recent QCD phase diagram (predicted by Fukushima-Hatsuda in 2010)



$\Rightarrow$ inhomogeneous chiral condensed phases could appear in real QCD

Typical shape of inhomogeneous chiral condensates

Flavor-SU(2) case — a general chiral order parameter: $\phi(z) \equiv \langle\bar\psi\psi\rangle(x) + i\langle\bar\psi i\gamma_5\tau_3\psi\rangle(x)$

- FF-type ($\phi_{\rm FF} = \Delta e^{iqz}$) ground state: 【Nakano-Tatsumi(2005); akin to Dautry-Nyman(1979)】

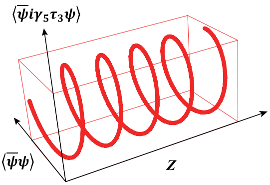
$$\langle\bar\psi\psi\rangle(z) = \Delta\cos(qz), \quad \langle\bar\psi i\gamma_5\tau_3\psi\rangle(z) = \Delta\sin(qz)$$

- LO-type ($\phi_{\rm LO} = \Delta(z)$) ground state: 【Nickel(2009); cf. Thies(2006)】

$$\langle\bar\psi\psi\rangle(z) = \Delta\sqrt{\nu}\,{\rm sn}(\Delta z|\nu)$$

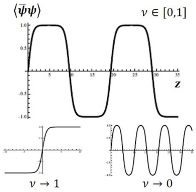
(Δ: amplitude, q: wavenumber, ν: elliptic modulus)

▷ Dual chiral density waves (DCDW)



(chiral spirals in 3+1D systems)

▷ Real kink crystals (RKC)



(periodic domain walls in 3+1D systems)

Outline

Basic features of 1D modulations

(mean-field results and fluctuation effects beyond MFA)

1D MODULATIONS (NJL RESULTS WITHIN MFA)

- NJL-model Lagrangian (chiral limit):

$$\mathcal{L}_{\mathsf{NJL}} = \bar{\psi} i \gamma^{\mu} \partial_{\mu} \psi + G \left[\left(\bar{\psi} \psi \right)^2 + \left(\bar{\psi} i \gamma_5 \tau_a \psi \right)^2 \right]$$

- MFA (condensates):

$$\sigma(x) \equiv \langle \overline{\psi}(x) \psi(x) \rangle, \quad \pi_a(x) \equiv \langle \overline{\psi}(x) i \gamma_5 \tau_a \psi(x) \rangle \delta_{a3}$$

- Gap equations (minimizing thermodynamic potential $\mathcal{V}$ w.r.t. σ, π) :

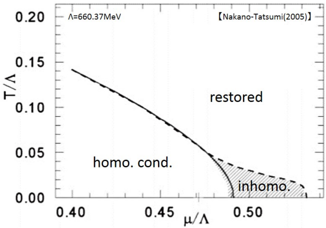
$$\frac{\delta \mathcal{V}_{\mathsf{MF}}(T, \mu; \sigma, \pi_a)}{\delta \sigma(x)} = \frac{\delta \mathcal{V}_{\mathsf{MF}}(T, \mu; \sigma, \pi_a)}{\delta \pi_a(x)} = 0$$

- ▷ difficulty in solving the eigenvalue eq. in 3+1D: $[i \partial\!\!\!/ + \sigma(x) + i \gamma_5 \tau_q \pi_a(x)] \psi = 0$
- ▷ using known 1+1D exact analytic solutions (and boosting transverse directions)

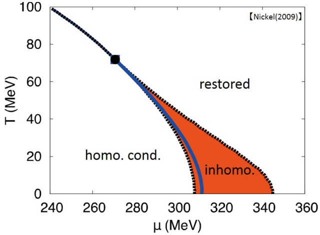
  thanks to a mathematical discovery of self-consistent solutions in 1+1D systems [Başar-Dunne-Thies(2009)]
- ▷ obtain gap solutions by minimizing $\mathcal{V}_{\mathrm{MF}}$ w.r.t. variational parameters (Δ, q, ν)

1D modulations (NJL results within MFA)

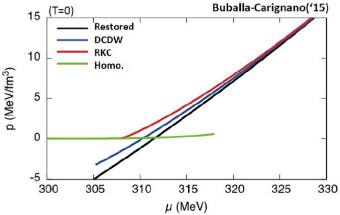
- DCDW: $\frac{\partial \mathcal{V}_{\text{MF}}(\phi_1)}{\partial \Delta, q} = 0$



- RKC: $\frac{\partial \mathcal{V}_{\text{MF}}(\phi_2)}{\partial \Delta, \nu} = 0$



- $T = 0$ results: free energies for DCDW and RKC condensates



$\Rightarrow$ RKC is energetically favored over DCDW within MFA in the chiral limit

$\Rightarrow$ qualitatively same even at $T > 0$

(but external magnetic fields turn the tables)

【cf. Frolov et al.(2010), Tatsumi et al.(2014)】

What if fluctuations are taken into account?

$$\phi(z) = \phi_0(z) + \delta\phi(z)$$

mean-fields fluctuations

1D modulations (Beyond mean-field treatments)

▷ DCDW ground state: $\phi_0 = (\Delta\cos qz, 0, 0, \Delta\sin qz)^T$

- Including fluctuations (NG modes β_i): $\phi = \phi_0 + \delta\phi$ 【Lee-Nakano-Tsue-Tatsumi-Friman;PRD(2015)】

$$\phi = \begin{pmatrix} \Delta\cos qz \\ 0 \\ 0 \\ \Delta\sin qz \end{pmatrix} + \begin{pmatrix} -\Delta\sin qz\beta_3 \\ \Delta\cos qz\beta_1 \\ \Delta\cos qz\beta_2 \\ \Delta\cos qz\beta_3 \end{pmatrix} \equiv \begin{pmatrix} \langle\sigma\rangle \\ 0 \\ 0 \\ \langle\pi_3\rangle \end{pmatrix} + \begin{pmatrix} \delta\sigma \\ \delta\pi_1 \\ \delta\pi_2 \\ \delta\pi_3 \end{pmatrix}$$

- Dispersion relations for NG modes

$$\omega_z^2 \propto 4q^2k_z^2 + (\vec{k}^2)^2 \qquad \text{for } \beta_3 \text{ (longitudinal mode)}$$

$$\omega_\perp^2 \propto 4q^2k_z^2 + (\vec{k}^2)^2 + \mathcal{O}(\vec{k}^6) \qquad \text{for } \beta_{1,2} \text{ (transverse modes)}$$

⇒ spatially anisotropic because of the lack of $\vec{k}_\perp^2$-term (akin to smectic liquid crystals)

⇒ which is due to the symmetry under rotations about transverse direction (slabs?)

- Impacts of NG modes

$$\langle\phi\rangle = \langle\phi_0 + \delta\phi\rangle \simeq \begin{pmatrix} \Delta\cos(qz)e^{-\sum_i\langle\beta_i^2\rangle/2} \\ 0 \\ 0 \\ \Delta\sin(qz)e^{-\langle\beta_3^2\rangle/2} \end{pmatrix} \xrightarrow{\text{IR}} 0 \quad \text{(washed out)}$$

where Gaussian fluctuations are logarithmically divergent at small k

$$\langle\beta_{1,2}^2\rangle \simeq \frac{1}{2\Delta^2}\int\frac{d^3k}{(2\pi)^3}\frac{T}{\omega_\perp^2} \xrightarrow{\text{IR}} \infty \quad \text{and} \quad \langle\beta_3^2\rangle \simeq \frac{1}{2\Delta^2}\int\frac{d^3k}{(2\pi)^3}\frac{T}{\omega_z^2} \xrightarrow{\text{IR}} \infty$$

1D modulations (Beyond mean-field treatments)

▷ Landau-Peierls instability

- DCDW phase is unstable due to thermal fluctuations: (NG modes at $T > 0$)

  $\langle \phi \rangle = 0$ (NG modes wash out long-range correlations)

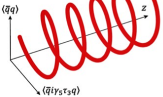
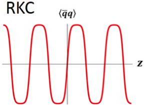
  $\Rightarrow$ but algebraic decay correlations remain (quasi-long-range order)

  $$\langle \phi(z\vec{e}_z) \cdot \phi^*(0) \rangle \sim \frac{1}{2}\Delta^2 \cos qz (z/z_0)^{-T/T_0}$$
  $$\langle \phi(x_\perp \vec{e}_\perp) \cdot \phi^*(0) \rangle \sim \frac{1}{2}\Delta^2 (x_\perp/x_0)^{-2T/T_0}$$
  [Lee-Nakano-Tsue-Tatsumi-Friman(2015)]

- Unlike disordered/normal phase with exponential decays, this phase exhibits algebraic decays, and thus could be realized as a quasi-one-dimensionally ordered phase, as in liquid crystals.
  [cf. Chaikin-Lubensky(2000)]

- RKC phase has the same result [cf. Hidaka-Kamikado-Kanazawa-Noumi(2015)]

<u>differences between DCDW and RKC</u>

	NG modes	Dispersion relations	Long-range correlations
DCDW 	3 (1 pi-ph coupled)	$\omega^2_{ph*pi_3} \sim Ak_z^2 + Bk_\perp^4$ $\omega^2_{pi_{1,2}} \sim A'k_z^2 + B'k_\perp^4$	$\langle \phi_i(x)\phi_j^*(0)\rangle \sim \begin{cases} \cos qz\,(L_z^{-\eta}) \\ L_\perp^{-\eta'} \end{cases}$
	different SSB	*spatially anisotropic (z: linear, xy: quadratic)*	*algebraic decays (QLRO)*
RKC 	4 (3 pions, 1 phonon)	$\omega^2_{ph} \sim ak_z^2 + bk_\perp^4$ $\omega^2_{pi} \sim a'k_z^2 + b'k_\perp^2$	$\langle \phi_i(x)\phi_j^*(0)\rangle \sim \begin{cases} \cos qz\,(L_z^{-\tilde\eta}) \\ L_\perp^{-\tilde\eta'} \end{cases}$

1D modulations (Beyond mean-field treatments)

▷ Possibilities inferred from Landau-Peierls theorem

- $T = 0$ limit (sufficiently low temperatures)

  $\langle\phi\rangle = \langle\phi_0 + \delta\phi\rangle \neq 0$ (ordered)

  ⇒ stable against quantum fluctuations due to convergent fluctuations $\langle\beta^2\rangle \propto \int d^3k\omega^{-1}$

- External magnetic fields

  $\omega^2 \sim ak_z^2 + b\vec{k}^2 + c(\vec{k}^2)^2$ for $B \neq 0$ (cf. $\omega^2 \sim ak_z^2 + c(\vec{k}^2)^2$ for $B = 0$)

  ⇒ dispersion of NG modes is modified
  (external magnetic fields explicitly break the rotational symmetry)

  ⇒ could be stabilized because of improved IR div

- Finite-size effects

  the system size as a IR cutoff ($\langle\beta^2\rangle$ converges; 1D structure remains if $\lambda \sim L_\perp$)

  ⇒ effectively mimic true LRO depending on finite-size effects (or experimental resolutions)

  【cf. Als-Nielsen *et al.*(1980); Baym-Friman-Grinstein(1982)】

- Two- and three-dimensional modulations

  similar suppression of Gaussian fluctuations can be expected for higher modulations

  ⇒ stabilization could occur

Beyond 1D modulations

Beyond 1D modulations (2D and 3D modulations)

▷ up to now only a few works have been devoted

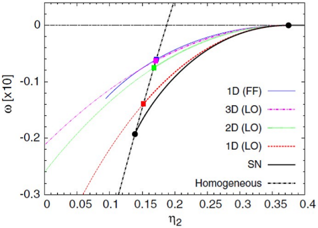
▷ there are no known analytic solutions for 2+1D or 3+1D systems (unlike 1+1D)

▷ assume some *ansätze* and compare their free energies with 1D condensate

---

- GL analysis at a Lifshitz point 【Abuki-Ishibashi-Suzuki(2012)】

  ⇒ using some *ansätze* (multidimensional LO-type real condensates), together with FF-type 1D



$$\phi_{\mathrm{FF;1D}}(\mathbf{x}) = \Delta\, e^{ikz}$$
$$\phi_{\mathrm{LO;1D}}(\mathbf{x}) = \sqrt{2}\,\Delta\,\sin(kz)$$
$$\phi_{\mathrm{LO;2D}}(\mathbf{x}) = \Delta(\sin(kx) + \sin(ky))$$
$$\phi_{\mathrm{LO;3D}}(\mathbf{x}) = \sqrt{\frac{2}{3}}\Delta(\sin(kx) + \sin(ky) + \sin(kz))$$

$$\boldsymbol{\Omega_{\mathrm{LO}} < \Omega_{\mathrm{FF}}} \quad , \quad \boldsymbol{\Omega_{\mathrm{1D}} < \Omega_{\mathrm{2D}} < \Omega_{\mathrm{3D}}}$$

(cf. for only FF-type free energies, $\Omega_{\mathrm{FF;2D}} < \Omega_{\mathrm{FF;1D}} < \Omega_{\mathrm{FF;3D}}$)

2D/3D is thermodynamically disfavored against 1D in the vicinity of LP

Beyond 1D modulations (2D and 3D modulations)

▷ up to now only a few works have been devoted

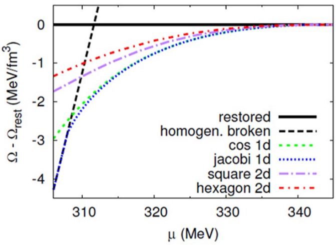
▷ there are no known analytic solutions for 2+1D or 3+1D systems (unlike 1+1D)

▷ assume some *ansätze* and compare their free energies with 1D condensate

---

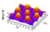
- NJL analysis at T=0 【Carignano-Buballa(2012)】

  ⇒ diagonalize a Hamiltonian for given *ansätze* (two-dimensional LO; square and hexagonal lattices)



Square lattice: 

$$\phi_{\mathrm{SQ;2D}}(x,y) = \Delta\cos(qx)\cos(qy)$$

Hexagonal lattice: 

$$\phi_{\mathrm{HEX;2D}}(x,y) = \frac{\Delta}{3}\left[2\cos(qx)\cos\left(\frac{1}{\sqrt{3}}qy\right) + \cos(\frac{2}{\sqrt{3}}qy)\right]$$

$$\mathbf{\Omega_{1D} < \Omega_{2D}}$$

2D is disfavored against 1D at $T = 0$ as well

Beyond 1D modulations (2D and 3D modulations)

▷ up to now only a few works have been devoted

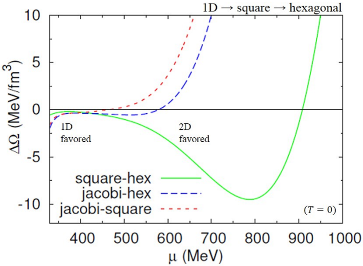
▷ there are no known analytic solutions for 2+1D or 3+1D systems (unlike 1+1D)

▷ assume some *ansätze* and compare their free energies with 1D condensate

---

▶ NJL analysis at T=0 【Carignano-Buballa(2012)】

⇒ the high-density side turns the tables (with modified parameter set)

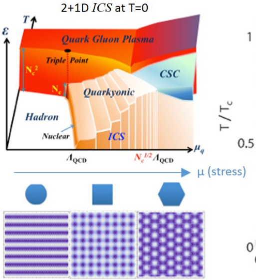


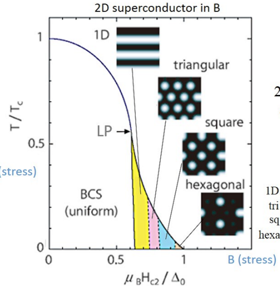
2D tends to be favored against 1D at higher densities

Beyond 1D modulations (2D and 3D modulations)

- some suggestions 【cf. Kojo et al.(2012) for ICS, Shimahara(1998);Machida-Shimahara(2007) for FFLO】

$\Rightarrow$ larger μ (or B) leads to a formation of multidimensional crystalline structures





2D has lower free energies than 1D at low T and high magnetic fields

1D: $\Delta(\mathbf{r}) = \Delta_1 e^{i\mathbf{q}\cdot\mathbf{r}} + \Delta_1 e^{-i\mathbf{q}\cdot\mathbf{r}} = 2\Delta_1 \cos(\mathbf{q}\cdot\mathbf{r})$

tri: $\Delta(\mathbf{r}) \propto \exp(i\mathbf{q}_1\cdot\mathbf{r}) + \exp(i\mathbf{q}_2\cdot\mathbf{r}) + \exp(i\mathbf{q}_3\cdot\mathbf{r})$

sq: $\Delta(\mathbf{r}) \propto \cos(qx) + \cos(qy)$,

hexa: $\Delta(\mathbf{r}) \propto \cos(\mathbf{q}_1\cdot\mathbf{r}) + \cos(\mathbf{q}_2\cdot\mathbf{r}) + \cos(\mathbf{q}_3\cdot\mathbf{r})$

$\Rightarrow$ there is a case that 3D is favored over 1D (in the context of pion condensation)

【Migdal-Markin-Mishustin(1976)】

№	Meson field	ξ
1	$\varphi_{\pi^0} = a(\cos kx + \cos ky + \cos kz)$	1
2	$\varphi_{\pi^0} = a(\cos kx + \cos ky)$	50/51
3	$\varphi_{\pi^0} = a \cos kx$	25/27
4	$\varphi_{\pi^0} = a \cos kx \cos ky \cos kz$	20/21

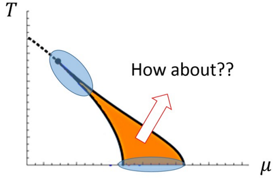
↑ favored

Beyond 1D modulations (2D and 3D modulations)

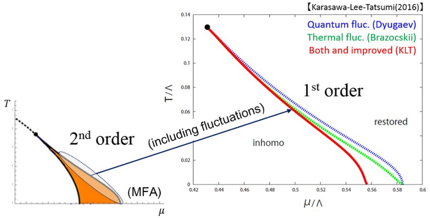
- possibilities

⇒ multidimensional chiral crystals may be possible in other areas

(we only know the areas near the LP and $T = 0$)



1D tends to be favored over 2D/3D
at least within MFA



Incidentally, fluctuation effects (beyond MFA) give rise to a modification of effective potential, which leads to a fluctuation-induced 1st-order transition (Brazovskii effect) if the system is stable. 【Brazovskii(1975);Dyugaev(1975);Karasawa-Lee-Tatsumi(2016), cf. Ohashi(2002)】

if so ⇒ easy to cause multidimensional structures?? (maybe nontrivial)

Beyond 1D modulations (2D and 3D modulations)

- toward a search for multidimensional structure
  - NJL model w/ RKC condensate:

$$\begin{cases} \mathcal{L}_{\rm MF} = \bar{\psi}\gamma^0 \left(i\partial_0 - H_D\right)\psi - \dfrac{\Delta(\mathbf{r})^2}{4G_s}, \qquad \Delta(\mathbf{r}) \equiv -2G_s\langle\bar{\psi}\psi\rangle(\mathbf{r}) \\ H_{D,\rm Weyl} = \begin{pmatrix} -i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla} & \Delta(\mathbf{r}) \\ \Delta(\mathbf{r}) & \boldsymbol{\sigma}\cdot\boldsymbol{\nabla} \end{pmatrix} \\ \Omega = -N_f N_c T \sum_{E_n} \int \dfrac{d^3\boldsymbol{p}}{(2\pi)^3} \ln\left(2\cosh\left(\dfrac{E_n-\mu}{2T}\right)\right) \end{cases}$$

  - it is necessary to know the quark energy spectrum E to obtain Ω
    → need to solve Dirac eq. $H_D\psi = E\psi$ for given $\Delta(\mathbf{r})$ (mass function)
    → but rather tough (simplification necessary for $\Delta(\mathbf{r})$)

  - we employ another procedure (giving density distributions, instead of $\Delta(\mathbf{r})$)
    → at fixed quark number density $\langle n\rangle = \frac{1}{V}\int d^3\mathbf{r}\, n(\mathbf{r})$, we search the lowest energy
    (*Thomas-Fermi approximation*)

Summary

Towards a multidimensional structure

- inhomogeneous chiral phases w/ 1D modulations:
  - within MFA and beyond
  - Landau-Peierls instability
- inhomogeneous chiral phases w/ 2D or 3D modulations:
  - 1D is favored at LP and $T = 0$
  - how about in other areas?
  - explore in other areas by numerical treatments w/ TFA (now ongoing)
- additional consideration should be considered:
  - β-equilibrium and charge neutrality (chiral pastas?)
  - nonvanishing μ_I, finite B, etc.

Thank you for your attention and patience!